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Time evolution, cyclic solutions and geometric phases for the generalized time-dependent harmonic oscillator

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Abstract
The generalized time-dependent harmonic oscillator is studied. Though several approaches to the solution of this model have been available, yet a new approach is presented here, which is very suitable for the study of cyclic solutions and geometric phases. In this approach, finding the time evolution operator for the Schrödinger equation is reduced to solving an ordinary differential equation for a c-number vector which moves on a hyperboloid in a three-dimensional space. Cyclic solutions do not exist for all time intervals. A necessary and sufficient condition for the existence of cyclic solutions is given. There may exist some particular time interval in which all solutions with definite parity, or even all solutions are cyclic. Criteria for the appearance of such cases are given. The known relation that the nonadiabatic geometric phase for a cyclic solution is proportional to the classical Hannay angle is reestablished. However, this is valid only for special cyclic solutions. For more general ones, the nonadiabatic geometric phase may contain an extra term. Several cases with relatively simple Hamiltonians are solved and discussed in detail. Cyclic solutions exist in most cases. The pattern of the motion, say, finite or infinite, cannot be simply determined by the nature of the Hamiltonian (elliptic or hyperbolic, etc.). For a Hamiltonian with a definite nature, the motion can change from one pattern to another, that is, some kind of phase transition may occur, if some parameter in the Hamiltonian goes through some critical value.

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1. Introduction

The harmonic oscillator is one of the most familiar models in physics. It is widely used in various fields, both classically and quantum mechanically. Several physical problems can be described by the harmonic oscillator with time-dependent parameters [1, 2]. More generally, such time-dependent parameters may describe approximately the interaction of the harmonic oscillator with some external degrees of freedom. Therefore, the time-dependent harmonic oscillator has been the subject of much theoretical research over decades [3–6]. Since the discovery of the geometric phase [7–14], the model has attracted more attention [15–22], because it is simple and serves as a good example for the study of geometric phases, just like the case of particles with spin and magnetic moment moving in time-dependent magnetic fields [23–37]. The two models are similar in that they involve similar Lie algebras. In fact, the Hamiltonian for spin in a magnetic field is an element of the $SO(3)$ algebra while that for the time-dependent harmonic oscillator is one of the $SO(2, 1)$ algebra. However, the time-dependent harmonic oscillator is of more interest since it has a classical counterpart, and the quantum motion can be compared with the classical one. For example, the relation between the geometric phase and the classical Hannay angle [38–40] is an interesting subject.

There exist mainly two approaches to the solution of the time-dependent harmonic oscillator. The main point of the first approach [3, 4] is to find an invariant operator and its eigenstates. The second approach employs time-dependent unitary transformations [6, 16]. Here we develop another approach to the solution of the Schrödinger equation. It is a further development of the approach previously used for spin moving in time-dependent magnetic fields [36]. First we find an invariant operator, and then go further to obtain the time evolution operator. It should be remarked that our method of finding the invariant operator is rather different from that in the first approach. In that approach the problem is reduced to solving a nonlinear differential equation. In our approach, it is reduced to solving a linear differential equation for a three-component c-number vector, and thus is simpler. On the other hand, the time-dependent unitary transformation approach in [16] seems even simpler than ours, but our approach can be easily generalized to other systems where the Hamiltonian is an element of a more complicated Lie algebra. However, the main advantage of our approach is that it is very suitable for the study of cyclic solutions and geometric phases.

In the literature there exists some argument that cyclic solutions are available for any time interval, say, $[0, \tau]$ where τ is arbitrary, because one can always choose the eigenstates of $U(\tau)$, the time evolution operator at τ, as initial conditions at $t = 0$. This is true. However, the problem is that, for the time-dependent harmonic oscillator, $U(\tau)$ may have no normalizable eigenstate. This is different from the case of spin, where no problem of normalizability has to be considered. Obviously, normalizable states are physically more interesting than nonnormalizable ones. Moreover, if one is interested in the geometric phase, it seems still unclear how to define it for the nonnormalizable cyclic solutions. The difficulty lies in the definition of the dynamical phase. If only normalizable states are to be considered, then one should be able to tell whether there are cyclic solutions for a given τ. This is not a trivial task even when the time evolution operator is explicitly available. In our approach, however, the problem can be solved naturally. We will give a necessary and sufficient condition for the existence of cyclic solutions in an arbitrarily given time interval.

It has been shown by several authors that the nonadiabatic geometric phase for a quantum cyclic state is equal to $-(n + 1/2)$ times the classical Hannay angle [19–21]. However, it does not seem very clear under what restriction on the initial condition this relation is valid, or whether modification of this relation is needed in some case where cyclic solutions exist for less restricted initial conditions. We will reestablish the above relation and show that it is valid only for cyclic solutions with special initial conditions. There exist several cases where more general cyclic solutions exist in some particular time interval. Among these cases two are of special interest. In one of the two cases all solutions are cyclic, and in the other case all solutions with definite parity are cyclic. We will give criteria for the appearance of such cases. In all these cases, the nonadiabatic geometric phase contains in general an extra term which depends on the initial condition, in addition to the one proportional to the classical Hannay angle. Similar situations have been found in other systems [35, 36, 41].

A normalizable state can be regarded as a wave packet in the configuration space. The motion of a wave packet can be roughly described by the change of its position and width. For the time-dependent harmonic oscillator, we will show that if the position of a wave packet is confined in a finite region, then its width also remains finite, and the reverse is also true. A problem that seems unclear concerns the relation between the pattern of the motion, say, finite or infinite, and the nature of the Hamiltonian, that is, elliptic, hyperbolic or critical (see section 2). In simple cases where the Hamiltonian is time independent (or the time dependence lies only in an overall factor), an elliptic Hamiltonian leads to finite motion and others lead to infinite motion. However, if the Hamiltonian is time dependent, the situation is complicated. We will see that for a Hamiltonian with a definite nature, say, elliptic, the motion of the wave packet may exhibit different patterns if some parameter in the Hamiltonian takes different values. In particular, when the parameter goes through some critical value, some kind of phase transition occurs, that is, the motion changes from one pattern to another. At the critical value, the motion has an independent pattern.

This paper is organized as follows. In section 2 we develop some mathematical formulae that will be used in the subsequent sections. In section 3 a new method is presented to derive the time evolution operator for the Schrödinger equation. In section 4 a necessary and sufficient condition for the existence of cyclic solutions in an arbitrarily given time interval $[0, \tau]$ is given, and the known relation between the nonadiabatic geometric phase and the classical Hannay angle is reestablished. In section 5 we study several cases where more cyclic solutions are available, and give criteria for the appearance of such cases. A modification to the above relation between the nonadiabatic geometric phase and the classical Hannay angle is also discussed in this section. In section 6 we study several examples where explicit results are available. The existence of cyclic solutions is discussed in detail, and it is seen that they exist in most of the examples. The evolution of normalizable states is also studied from the point of view of wave packets. Various patterns of motion are revealed in these examples, and phase transition is explicitly observed. Section 7 is devoted to a brief summary. In the appendix we briefly discuss how to extend our formalism to a more general system.

2. The model and some mathematics

The time-dependent harmonic oscillator is described by the Schrödinger equation

$$\mathrm{i}\hbar\partial_t\psi(t) = H(t)\psi(t) \tag{1a}$$

with the following Hamiltonian.

$$H(t) = \omega(t)\boldsymbol{K}\cdot\boldsymbol{n}^g(t) \tag{1b}$$

where $\omega(t)$ is a time-dependent frequency parameter, $\boldsymbol{n}(t) = (n_1(t), n_2(t), n_3(t))$ is a time-dependent c-number vector and $\boldsymbol{K} = (K_1, K_2, K_3)$ is an operator vector defined below. In this paper, any vector $\boldsymbol{a} = (a_1, a_2, a_3)$ has an associated vector $\boldsymbol{a}^g = (-a_1, -a_2, a_3)$, and the scalar product between two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ always appears as $(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{a}\cdot\boldsymbol{b}^g = \boldsymbol{a}^g\cdot\boldsymbol{b} = a_3b_3 - a_1b_1 - a_2b_2$. In matrix form these are $a^g = ga$ and $(\boldsymbol{a}, \boldsymbol{b}) = a^{\mathrm{t}}gb = b^{\mathrm{t}}ga$, where a and b are now column vectors and the superscript 't' denotes transposition, and $g = \mathrm{diag}\ (-1, -1, 1)$. In the following the square of a vector $\boldsymbol{a}^2$ always stands for $\boldsymbol{a} \cdot \boldsymbol{a}^g$ rather than $\boldsymbol{a} \cdot \boldsymbol{a}$. In the above Hamiltonian we always choose ω such that $\boldsymbol{n}^2$ equals 1, -1 or 0. Thus among the components of $\boldsymbol{n}$ only two are independent, and the Hamiltonian involves three independent parameters. The vector $\boldsymbol{K}$ is defined as

$$K_1 = \frac{1}{2}\left(\mu_0\omega_0 X^2 - \frac{P^2}{\mu_0\omega_0}\right) \qquad K_2 = -\frac{1}{2}(XP + PX) \qquad K_3 = \frac{1}{2}\left(\mu_0\omega_0 X^2 + \frac{P^2}{\mu_0\omega_0}\right) \tag{2a}$$

where X is the coordinate and P is the momentum, satisfying $[X, P] = \mathrm{i}\hbar$, μ_0 is the mass of the particle and ω_0 is some constant frequency parameter. For simplicity in notation, we define $x = \sqrt{\mu_0\omega_0}X$ and $p = P/\sqrt{\mu_0\omega_0}$, which still satisfy the commutation relation $[x, p] = \mathrm{i}\hbar$, then the above expressions become

$$K_1 = \tfrac{1}{2}(x^2 - p^2) \qquad K_2 = -\tfrac{1}{2}(xp + px) \qquad K_3 = \tfrac{1}{2}(x^2 + p^2). \tag{2b}$$

Note that both x and p have the same dimensionality as $\sqrt{\hbar}$, and $\boldsymbol{K}$ has the same dimensionality as $\hbar$. The components of $\boldsymbol{K}$ satisfy the commutation relation

$$[K_i, K_j] = -2\mathrm{i}\hbar\epsilon_{ijk}K_k^g. \tag{3}$$

An equivalent form is $\left[K_i^g, K_j^g\right] = -2\mathrm{i}\hbar\epsilon_{ijk}K_k$. This is an $SO(2,1)$ algebra. The Hamiltonian is said to be elliptic, hyperbolic or critical if $\boldsymbol{n}^2$ equals 1, -1 or 0, respectively.

To solve the Schrödinger equation we need some operator formulae. We will briefly derive them here. Let

$$\boldsymbol{F}(\xi) = Q(\xi, \boldsymbol{b})\boldsymbol{K}Q^\dagger(\xi, \boldsymbol{b}) \qquad Q(\xi, \boldsymbol{b}) = \exp\left(-\tfrac{1}{2}\,\mathrm{i}\hbar^{-1}\xi\boldsymbol{K} \cdot \boldsymbol{b}^g\right) \tag{4}$$

where ξ is a real parameter and $\boldsymbol{b}$ is a real vector independent of ξ. This is a unitary transformation of $\boldsymbol{K}$. Using equation (3) it can be shown that

$$\boldsymbol{F}'(\xi) = \boldsymbol{b}^g \times \boldsymbol{F}^g(\xi) \tag{5}$$

where the prime indicates a derivative with respect to ξ. An equivalent equation is $\boldsymbol{F}^{g\prime}(\xi) = \boldsymbol{b} \times \boldsymbol{F}(\xi)$, which is convenient in obtaining

$$\boldsymbol{F}''(\xi) + \boldsymbol{b}^2\boldsymbol{F}(\xi) = [\boldsymbol{F}(\xi) \cdot \boldsymbol{b}^g]\boldsymbol{b}. \tag{6}$$

However, equation (5) leads to $\boldsymbol{F}'(\xi) \cdot \boldsymbol{b}^g = 0$, and thus $\boldsymbol{F}(\xi) \cdot \boldsymbol{b}^g = \boldsymbol{F}(0) \cdot \boldsymbol{b}^g = \boldsymbol{K} \cdot \boldsymbol{b}^g$. Therefore equation (6) is simplified as

$$\boldsymbol{F}''(\xi) + \boldsymbol{b}^2\boldsymbol{F}(\xi) = (\boldsymbol{K} \cdot \boldsymbol{b}^g)\boldsymbol{b}. \tag{7a}$$

This is a simple equation. With the initial condition

$$\boldsymbol{F}(0) = \boldsymbol{K} \qquad \boldsymbol{F}'(0) = \boldsymbol{b}^g \times \boldsymbol{K}^g \tag{7b}$$

the solution is easily fixed. The results are listed below.

If $\boldsymbol{b}^2 = 1$, we denote $\boldsymbol{b}$ by $\boldsymbol{b}_+$, and have

$$Q(\xi, \boldsymbol{b}_+)\boldsymbol{K}Q^\dagger(\xi, \boldsymbol{b}_+) = \left[\boldsymbol{K} - \left(\boldsymbol{K} \cdot \boldsymbol{b}_+^g\right)\boldsymbol{b}_+\right]\cos\xi + \boldsymbol{b}_+^g \times \boldsymbol{K}^g \sin\xi + \left(\boldsymbol{K} \cdot \boldsymbol{b}_+^g\right)\boldsymbol{b}_+. \tag{8}$$

When $\xi = 2N\pi$ where N is an integer, we have $Q(2N\pi, \boldsymbol{b}_+)\boldsymbol{K}Q^\dagger(2N\pi, \boldsymbol{b}_+) = \boldsymbol{K}$, or $Q(2N\pi, \boldsymbol{b}_+)\boldsymbol{K} = \boldsymbol{K}Q(2N\pi, \boldsymbol{b}_+)$. That is, $Q(2N\pi, \boldsymbol{b}_+)$, and in particular, $\exp(\mathrm{i}\hbar^{-1}N\pi K_3)$, commutes with $\boldsymbol{K}$. However, this does not mean that $Q(2N\pi, \boldsymbol{b}_+)$ is a c-number, since a c-number must commute with x and p. But we will see below that $Q(4N\pi, \boldsymbol{b}_+)$ is indeed a c-number.

If $\boldsymbol{b}^2 = -1$, we denote $\boldsymbol{b}$ by $\boldsymbol{b}_-$, and have

$$Q(\xi, \boldsymbol{b}_-)\boldsymbol{K}Q^\dagger(\xi, \boldsymbol{b}_-) = [\boldsymbol{K} + (\boldsymbol{K} \cdot \boldsymbol{b}_-^g)\boldsymbol{b}_-]\cosh\xi + \boldsymbol{b}_-^g \times \boldsymbol{K}^g \sinh\xi - (\boldsymbol{K} \cdot \boldsymbol{b}_-^g)\boldsymbol{b}_-. \tag{9}$$

When $\boldsymbol{b}_- = (-\sin\phi, \cos\phi, 0)$, the result is useful in the following sections. In this case we denote $\boldsymbol{b}_-$ by $\boldsymbol{b}_\phi$ and $Q(\xi, \boldsymbol{b}_-)$ by $Q(\xi, \phi)$:

$$Q(\xi,\phi) = \exp\left(-\tfrac{1}{2}\,\mathrm{i}\hbar^{-1}\xi\boldsymbol{K}\cdot\boldsymbol{b}_\phi^g\right) = \exp\left[-\tfrac{1}{2}\,\mathrm{i}\hbar^{-1}\xi(K_1\sin\phi - K_2\cos\phi)\right]. \tag{10}$$

In particular we write down

$$Q(\xi,\phi)K_3Q^\dagger(\xi,\phi) = K_3\cosh\xi - K_1\sinh\xi\cos\phi - K_2\sinh\xi\sin\phi. \tag{11}$$

If $\boldsymbol{b}^2 = 0$, we denote $\boldsymbol{b}$ by $\boldsymbol{b}_0$, and have

$$Q(\xi,\boldsymbol{b}_0)\boldsymbol{K}Q^\dagger(\xi,\boldsymbol{b}_0) = \boldsymbol{K} + \xi\boldsymbol{b}_0^g\times\boldsymbol{K}^g + \tfrac{1}{2}\xi^2\left(\boldsymbol{K}\cdot\boldsymbol{b}_0^g\right)\boldsymbol{b}_0. \tag{12}$$

For a c-number vector $\boldsymbol{a}$, one can define a transformed vector $\boldsymbol{A}$ by

$$\boldsymbol{K}\cdot\boldsymbol{A} = Q(\xi,\boldsymbol{b})\boldsymbol{K}\cdot\boldsymbol{a}Q^\dagger(\xi,\boldsymbol{b}). \tag{13}$$

It can be shown by straightforward calculations that $\boldsymbol{A}^2 = \boldsymbol{a}^2$, regardless of the values of ξ and $\boldsymbol{b}$. Thus the elements of the $SO(2,1)$ algebra, $\boldsymbol{K}\cdot\boldsymbol{a}$, are distinguished into three classes, characterized by whether $\boldsymbol{a}^2$ is positive, negative or zero. They cannot be connected by any one of the above unitary transformations. For example, K_3 cannot be transformed to K_1 and vice versa. Geometrically, the surface defined by $\boldsymbol{a}^2 = 0$ is a cone in the $\boldsymbol{a}$ space. Therefore, the three classes of elements are characterized by whether $\boldsymbol{a}$ is inside, outside or on the cone.

Similarly, one can calculate the unitary transformation of x and p. Here we only write down the result for the case with $\boldsymbol{b}^2 = 1$ (the subscript '+' in the components of $\boldsymbol{b}_+$ is omitted).

$$Q(\xi,\boldsymbol{b}_+)xQ^\dagger(\xi,\boldsymbol{b}_+) = x\left(\cos\frac{\xi}{2} - b_2\sin\frac{\xi}{2}\right) - p(b_3+b_1)\sin\frac{\xi}{2} \tag{14a}$$

$$Q(\xi,\boldsymbol{b}_+)pQ^\dagger(\xi,\boldsymbol{b}_+) = x(b_3-b_1)\sin\frac{\xi}{2} + p\left(\cos\frac{\xi}{2} + b_2\sin\frac{\xi}{2}\right). \tag{14b}$$

Here we see that $Q(4N\pi, \boldsymbol{b}_+)$ commutes with x and p. Thus it must be a c-number, as mentioned above.

The final point of this section concerns the eigenvalues and eigenstates of the operator $\boldsymbol{K}$. We are only interested in normalizable, or bound states. K_1 and K_2 do not have normalizable eigenstates. The eigenvalues of K_3 are $k_n\hbar$ where $k_n = n + 1/2$ and $n = 0, 1, 2, \ldots$. The corresponding eigenstates will be denoted by ψ_n. In the coordinate representation, $\psi_n(x) = N_n\exp(-x^2/2\hbar)H_n(x/\sqrt{\hbar})$, where H_n are Hermite polynomials and $N_n = (1/2^n n!\sqrt{\pi\hbar})^{1/2}$. If a vector $\boldsymbol{e}$ satisfies $\boldsymbol{e}^2 > 0$, then $\boldsymbol{K}\cdot\boldsymbol{e}^g$ has normalizable eigenstates. Without loss of generality, we take $\boldsymbol{e}^2 = 1$ and $e_3 > 0$, then $\boldsymbol{e}$ can be written as

$$\boldsymbol{e} = (\sinh\xi\cos\phi, \sinh\xi\sin\phi, \cosh\xi). \tag{15}$$

According to equation (11), $\boldsymbol{K}\cdot\boldsymbol{e}^g = Q(\xi,\phi)K_3Q^\dagger(\xi,\phi)$. Therefore the eigenvalues of $\boldsymbol{K}\cdot\boldsymbol{e}^g$ are still $k_n\hbar$, and the corresponding eigenstates are

$$\psi_n^e = Q(\xi,\phi)\psi_n = \exp\left(-\tfrac{1}{2}\,\mathrm{i}\hbar^{-1}\xi\boldsymbol{K}\cdot\boldsymbol{b}_\phi^g\right)\psi_n. \tag{16}$$

This result will be employed below.

3. Time evolution operator

In this section we deal with the time evolution operator for the Schrödinger equation (1). We define a time-dependent c-number vector $\boldsymbol{e}(t)$ by the differential equation

$$\dot{\boldsymbol{e}}(t) = -2\omega(t)\boldsymbol{n}^g(t)\times\boldsymbol{e}^g(t) \tag{17a}$$

where the overdot denotes differentiation with respect to t, and the initial condition

$$\boldsymbol{e}(0) = \boldsymbol{e}_0 \tag{17b}$$

where $\boldsymbol{e}_0$ satisfies

$$\boldsymbol{e}_0^2 = 1 \qquad e_{03} > 0 \tag{17c}$$

and is otherwise arbitrary. We would assume that $\boldsymbol{n}(t)$ and $\omega(t)$ vary continuously, so that any solution $\boldsymbol{e}(t)$ is well behaved. If one solution to this equation can be found, then the time evolution operator for the Schrödinger equation can be worked out. It should be remarked that the above equation for $\boldsymbol{e}(t)$ is the one satisfied by $\langle \boldsymbol{K} \rangle$, the mean value of the operator vector $\boldsymbol{K}$ in an arbitrary state (cf equation (27) below). More discussions can be found in the appendix.

We take the initial state of the system to be $\psi(0) = \psi_n^{e_0}$, that is

$$\boldsymbol{K} \cdot \boldsymbol{e}_0^g \psi(0) = k_n \hbar \psi(0) \qquad n = 0, 1, 2, \ldots. \tag{18}$$

If ψ evolves according to equation (1) and $\boldsymbol{e}$ evolves according to equation (17), then

$$\boldsymbol{K} \cdot \boldsymbol{e}^g(t) \psi(t) = k_n \hbar \psi(t) \tag{19}$$

would hold at all later times. This can be easily proved by induction.

By definition, equation (19) is valid at $t = 0$. We assume that it is valid at time t, what we need to do is to show that it is also true at time $t + \Delta t$, where Δt is an infinitesimal increment of time. In fact, using equations (1) and (17) we have

$$\psi(t + \Delta t) = \psi(t) - \mathrm{i}\hbar^{-1} \omega(t) \boldsymbol{K} \cdot \boldsymbol{n}^g(t) \psi(t) \Delta t \tag{20a}$$

$$\boldsymbol{e}(t + \Delta t) = \boldsymbol{e}(t) - 2\omega(t) \boldsymbol{n}^g(t) \times \boldsymbol{e}^g(t) \Delta t. \tag{20b}$$

After some simple algebra, the conclusion is achieved.

It should be remarked here that $\boldsymbol{K} \cdot \boldsymbol{e}^g(t)$ is an invariant operator, so that it has time-independent eigenvalues. Indeed, it is easy to show that

$$\mathrm{i}\hbar \boldsymbol{K} \cdot \dot{\boldsymbol{e}}^g(t) + [\boldsymbol{K} \cdot \boldsymbol{e}^g(t), H] = 0. \tag{21}$$

We see that once a solution $\boldsymbol{e}(t)$ is found, an invariant operator is obtained. Since $\boldsymbol{e}(t)$ satisfies a linear differential equation, this method is convenient. Moreover, it can be easily generalized to systems where the Hamiltonian is an element of a more complicated Lie algebra. A brief discussion on this point is given in the appendix.

From equation (17) it is easy to show that $\boldsymbol{e}^2(t) = \boldsymbol{e}_0^2 = 1$. This yields $|e_3(t)| \geqslant 1$. As is assumed, $\boldsymbol{e}(t)$ varies continuously, thus $e_3(t)$ keeps its original sign at all later times. Therefore, $\boldsymbol{e}(t)$ can be written in the form of equation (15), where $\xi = \xi(t)$ and $\phi = \phi(t)$, and

$$\psi(t) = \exp[\mathrm{i}\alpha_n(t)] Q(\xi(t), \phi(t)) \psi_n \tag{22}$$

where $\alpha_n(t)$ is a phase that cannot be determined by the eigenvalue equation. However, $\alpha_n(t)$ is not arbitrary. To satisfy the Schrödinger equation, it should be determined by the other variables $\xi(t)$ and $\phi(t)$. In fact, the above equation yields

$$(\psi(t), \psi(t + \Delta t)) = 1 + \mathrm{i}\dot{\alpha}_n(t) \Delta t + (\psi_n, Q^\dagger(\xi, \phi) \partial_t Q(\xi, \phi) \psi_n) \Delta t. \tag{23}$$

Using the formula [42]

$$\mathrm{e}^{F(t)} \partial_t \, \mathrm{e}^{-F(t)} = -\int_0^1 \mathrm{e}^{\lambda F(t)} \dot{F}(t) \, \mathrm{e}^{-\lambda F(t)} \, \mathrm{d}\lambda \tag{24}$$

where $F(t)$ is any operator depending on t, then using equation (9), and noting that $(\psi_n, K_1 \psi_n) = (\psi_n, K_2 \psi_n) = 0$, we obtain

$$(\psi(t), \psi(t + \Delta t)) = 1 + \mathrm{i}\dot{\alpha}_n(t) \Delta t - \tfrac{1}{2} \mathrm{i} k_n \dot{\phi}(t) [\cosh \xi(t) - 1] \Delta t. \tag{25}$$

On the other hand, from equation (20) we have

$$(\psi(t), \psi(t+\Delta t)) = 1 - \mathrm{i}\hbar^{-1}\omega(t)\boldsymbol{u}(t)\cdot\boldsymbol{n}^g(t)\Delta t \tag{26}$$

where

$$\boldsymbol{u}(t) = (\psi(t), \boldsymbol{K}\psi(t)). \tag{27}$$

This definition will be repeatedly used below. It is easy to show that $\boldsymbol{u}(t)$ satisfies the same equation as $\boldsymbol{e}(t)$, and for the above initial state $\boldsymbol{u}(0) = k_n\hbar\boldsymbol{e}_0$, so we have $\boldsymbol{u}(t) = k_n\hbar\boldsymbol{e}(t)$. Comparing the two results above and taking this relation into account, we obtain

$$\dot{\alpha}_n(t) = \tfrac{1}{2}k_n\dot{\phi}(t)[\cosh\xi(t) - 1] - k_n\omega(t)\boldsymbol{e}(t)\cdot\boldsymbol{n}^g(t). \tag{28}$$

Therefore

$$\alpha_n(t) - \alpha_n(0) = k_n\alpha(t) \tag{29}$$

where

$$\alpha(t) = \frac{1}{2}\int_0^t \dot{\phi}(t')[\cosh\xi(t') - 1]\,\mathrm{d}t' - \int_0^t \omega(t')\boldsymbol{e}(t')\cdot\boldsymbol{n}^g(t')\,\mathrm{d}t'. \tag{30}$$

Substituting into equation (22) we obtain

$$\psi(t) = Q(\xi(t), \phi(t))\exp[\mathrm{i}\hbar^{-1}\alpha(t)K_3]Q^\dagger(\xi(0), \phi(0))\psi(0). \tag{31}$$

We denote the time evolution operator as $U(t)$, defined by the equation $\psi(t) = U(t)\psi(0)$ with an arbitrary $\psi(0)$, then the above equation is equivalent to

$$U(t)\psi_n^{e_0} = Q(\xi(t), \phi(t))\exp[\mathrm{i}\hbar^{-1}\alpha(t)K_3]Q^\dagger(\xi(0), \phi(0))\psi_n^{e_0}. \tag{32}$$

Now an arbitrary normalizable initial state $\psi(0)$ can be expanded as

$$\psi(0) = \sum_n c_n\psi_n^{e_0}. \tag{33}$$

Applying $U(t)$ to both sides of this equation, using equation (32), and noting that the operators on the right-hand side of that equation are independent of n, we immediately realize that equation (31) is in fact valid for an arbitrary initial state. Thus we arrive at the result

$$U(t) = \exp\left[-\tfrac{1}{2}\mathrm{i}\hbar^{-1}\xi(t)\boldsymbol{K}\cdot\boldsymbol{b}_\phi^g(t)\right]\exp[\mathrm{i}\hbar^{-1}\alpha(t)K_3]\exp\left[\tfrac{1}{2}\mathrm{i}\hbar^{-1}\xi(0)\boldsymbol{K}\cdot\boldsymbol{b}_\phi^g(0)\right]. \tag{34a}$$

Using equation (9), it can be recast in the form

$$U(t) = \exp\left[-\tfrac{1}{2}\mathrm{i}\hbar^{-1}\xi(t)\boldsymbol{K}\cdot\boldsymbol{b}_\phi^g(t)\right]\exp\left[\tfrac{1}{2}\mathrm{i}\hbar^{-1}\xi(0)\boldsymbol{K}\cdot\boldsymbol{b}_\phi^g(0)\right]\exp\left[\mathrm{i}\hbar^{-1}\alpha(t)\boldsymbol{K}\cdot\boldsymbol{e}_0^g\right]. \tag{34b}$$

Equation (34*b*) is suitable for the general discussions below, while equation (34*a*) may be more convenient for practical calculations.

Let us make some remarks on the result. First, we see that once a solution of equation (17) is found, the time evolution operator for equation (1) is available. The result depends formally on $\boldsymbol{e}_0$, but $\boldsymbol{e}_0$ is merely an auxiliary object, hence the result must be essentially independent of it, though it might be difficult to prove this explicitly. On the other hand, it is the flexibility in the choice of $\boldsymbol{e}_0$ that makes it convenient for the general discussions of cyclic solutions. In practical calculations, one should choose a solution $\boldsymbol{e}(t)$ that is as simple as possible such that $U(t)$ can be easily reduced to the simplest form. Second, the operator $U(t)$ depends not only on $\boldsymbol{e}(t)$, but also on the history of it. This is obvious from equation (30). Third, though $\phi(t)$ is indefinite when $\xi(t) = 0$, it is obvious that $U(t)$ is well behaved everywhere. Fourth, by straightforward calculations it can be shown that $\mathrm{i}\hbar\partial_t U(t) = \omega(t)\boldsymbol{K}\cdot\boldsymbol{n}^g(t)U(t)$ and $U(0) = 1$, as expected. In other words, though $U(t)$ is obtained by considering the evolution of normalizable states, it is also valid for nonnormalizable ones.

4. Cyclic solutions and geometric phases

Now we can go further to discuss cyclic solutions in any time interval $[0, \tau]$, where τ is an arbitrarily given time. These cyclic solutions are not necessarily cyclic in subsequent time intervals with the same length, say, $[\tau, 2\tau]$.

Since equation (17) is a linear differential equation, the general solution $\boldsymbol{e}(t)$ must depend on the initial vector $\boldsymbol{e}_0$ linearly. Thus it can be written in a matrix form

$$e(t) = E(t)e_0 \tag{35}$$

where $e(t)$ and e_0 are column vectors, and $E(t)$ is a 3×3 matrix which is obviously real. If both $\boldsymbol{e}^{(1)}(t)$ and $\boldsymbol{e}^{(2)}(t)$ are solutions to equation (17), it is easy to show that $\boldsymbol{e}^{(1)}(t) \cdot \boldsymbol{e}^{(2)g}(t) = \boldsymbol{e}^{(1)}(0) \cdot \boldsymbol{e}^{(2)g}(0)$. Therefore the matrix $E(t)$ satisfies

$$E^{\mathrm{t}}(t)gE(t) = g. \tag{36}$$

This yields $\det E(t) = \pm 1$. As is assumed, $E(t)$ varies continuously, and $\det E(0) = 1$, so that $\det E(t) = 1$. Therefore the product of the three eigenvalues of $E(t)$ must be 1, and none can be zero. Now if e_σ is an eigenvector with eigenvalue σ, that is, $Ee_\sigma = \sigma e_\sigma$, it can be easily shown that $E^{\mathrm{t}}(ge_\sigma) = \sigma^{-1}(ge_\sigma)$. This means that σ^{-1} is an eigenvalue of E^{t}, and thus an eigenvalue of E. Therefore the eigenvalues of $E(t)$ should be $\{1, \sigma(t), \sigma^{-1}(t)\}$. Since E is real, σ^* is its eigenvalue if σ is one. Therefore, if $\sigma(t)$ is complex, it must be unit: $|\sigma(t)| = 1$.

If $\sigma(\tau) \neq 1$ at the time τ, one eigenvector $\boldsymbol{\eta}(\tau)$ of the matrix $E(\tau)$ with eigenvalue 1 can be found, which satisfies $E(\tau)\eta(\tau) = \eta(\tau)$. It can be taken as real. $\boldsymbol{\eta}^2(\tau)$ may be positive, negative or zero, depending on $E(\tau)$ and τ. First we consider the case with

$$\boldsymbol{\eta}^2(\tau) > 0. \tag{37}$$

Because $\boldsymbol{\eta}(\tau)$ is only determined up to a constant factor, we can choose that constant such that $\boldsymbol{\eta}^2(\tau) = 1$ and $\eta_3 > 0$. Then we can take

$$\boldsymbol{e}_0 = \boldsymbol{\eta}(\tau) \tag{38}$$

as the initial condition in equation (17), and have $e(\tau) = E(\tau)e_0 = E(\tau)\eta(\tau) = \eta(\tau) = e_0$, that is

$$\boldsymbol{e}(\tau) = \boldsymbol{e}_0. \tag{39}$$

This means that $\xi(\tau) = \xi(0)$ and $\boldsymbol{b}_\phi(\tau) = \boldsymbol{b}_\phi(0)$, and leads to

$$U(\tau) = \exp\left[\mathrm{i}\hbar^{-1}\alpha(\tau)\boldsymbol{K} \cdot \boldsymbol{e}_0^g\right]. \tag{40}$$

Now it is clear that with the initial condition $\psi(0) = \psi_n^{e_0}$ $(n = 0, 1, 2, \ldots)$, we have a cyclic solution in the time interval $[0, \tau]$. More specifically, $\psi(\tau) = \mathrm{e}^{\mathrm{i}\delta_n}\psi(0)$, where the total phase change is $\delta_n = k_n\alpha(\tau)$, mod 2π, with $\alpha(\tau)$ given by

$$\alpha(\tau) = \frac{1}{2}\int_0^\tau \dot{\phi}(t)[\cosh\xi(t) - 1]\,\mathrm{d}t - \int_0^\tau \omega(t)\boldsymbol{e}(t) \cdot \boldsymbol{n}^g(t)\,\mathrm{d}t. \tag{41}$$

For the present state, $\boldsymbol{u}(t) = k_n\hbar\boldsymbol{e}(t)$, so the dynamic phase $\beta_n = -\hbar^{-1}\int_0^\tau \langle H(t)\rangle\,\mathrm{d}t$ turns out to be

$$\beta_n = -k_n\int_0^\tau \omega(t)\boldsymbol{e}(t) \cdot \boldsymbol{n}^g(t)\,\mathrm{d}t. \tag{42}$$

Therefore the nonadiabatic geometric phase $\gamma_n = \delta_n - \beta_n$ is given by

$$\gamma_n = -k_n\Delta\theta_{\mathrm{g}} \qquad \mathrm{mod}\ 2\pi \tag{43}$$

where

$$\Delta\theta_{\rm g} = -\frac{1}{2}\int_0^\tau \dot{\phi}(t)[\cosh\xi(t) - 1]\,{\rm d}t = -\frac{1}{2}\int_0^\tau \frac{e_1(t)\dot{e}_2(t) - \dot{e}_1(t)e_2(t)}{e_3(t)+1}\,{\rm d}t \tag{44}$$

will be shown to be the classical Hannay angle below. Because $\boldsymbol{e}^2(t) = 1$ and $e_3(t) > 0$, $\boldsymbol{e}(t)$ moves on the upper sheet of a hyperboloid. This is a basic consequence of the fact that the Hamiltonian is an element of the $SO(2,1)$ algebra. The above integral can be recast in two other forms

$$\Delta\theta_{\rm g} = \mp\frac{1}{2}\int_S \frac{{\rm d}S}{\sqrt{e_1^2+e_2^2+e_3^2}} = \mp\frac{1}{2}\int_{S_{12}} \frac{{\rm d}S_{12}}{\sqrt{1+e_1^2+e_2^2}} \tag{45}$$

where S is the surface enclosed by the closed trace of $\boldsymbol{e}(t)$ on the hyperboloid, and dS the surface element; S_{12} is the projection of S on the e_1e_2 plane, and dS_{12} the area element; the upper (lower) sign corresponds to an anticlockwise (clockwise) trace of $\boldsymbol{e}(t)$. The geometric nature of the Hannay angle is obvious from the above expression, because it depends only on the closed trace of $\boldsymbol{e}(t)$, but not on the details of the traversing process. Because of the relation (43), the geometric nature of the nonadiabatic geometric phase is also obvious.

Thus equation (37) is a sufficient condition for the existence of cyclic solutions. Under this condition there exists at least a denumerable set of normalizable cyclic solutions in the time interval $[0,\tau]$. Of course, they may be trivial ones in some cases. All phases can be expressed in terms of the vector $\boldsymbol{e}(t)$. The relation between the nonadiabatic geometric phase and the Hannay angle is reestablished.

States with initial condition other than the above ones are in general not cyclic, even though in these initial states $\boldsymbol{u}(0)$ points in the direction of $\boldsymbol{e}_0$ such that $\boldsymbol{u}(\tau) = \boldsymbol{u}(0)$. However, if $\alpha(\tau)/\pi$ happens to be a rational number, more cyclic solutions are available, and the above relation between the nonadiabatic geometric phase and the Hannay angle would need modification for these cyclic solutions. This will be discussed in the next section.

If $\boldsymbol{\eta}^2(\tau) \leqslant 0$, one can still take $\boldsymbol{e}(0) = \boldsymbol{\eta}(\tau)$ as the initial condition for equation (17a), and have $\boldsymbol{e}(\tau) = \boldsymbol{e}(0)$. However, this solution cannot be used in the time evolution operator (34a), (34b) and no similar discussions to the above are available. In fact, there is no normalizable cyclic solution in this case, since the condition (37) is also a necessary one. This is proved below.

If there exists one normalizable cyclic solution in the time interval $[0,\tau]$, that is, $\psi(\tau) = {\rm e}^{{\rm i}\delta}\psi(0)$, then in this state $\boldsymbol{u}(\tau) = \boldsymbol{u}(0)$, or $u(\tau) = u(0)$ in the form of column vectors. As pointed out before, $\boldsymbol{u}(t)$ satisfies the same equation as $\boldsymbol{e}(t)$, thus $u(\tau) = E(\tau)u(0)$. Comparing the two relations we obtain $E(\tau)u(0) = u(0)$. In other words, $u(0)$ is an eigenvector of $E(\tau)$ with eigenvalue 1. The remaining point is to show that $\boldsymbol{u}^2(0) > 0$. Indeed, for any normalizable state $\psi(t)$, it is not difficult to show that $\boldsymbol{u}^2(t) \geqslant \hbar^2/4 > 0$, by using the Schwarz inequality.

To conclude this section let us work out the Hannay angle in terms of $\boldsymbol{e}(t)$. We denote the classical coordinate by $q_{\rm c}$ and momentum by $p_{\rm c}$, and define a vector $\boldsymbol{I}$ as

$$I_1 = \tfrac{1}{2}\left(q_{\rm c}^2 - p_{\rm c}^2\right) \qquad I_2 = -q_{\rm c}p_{\rm c} \qquad I_3 = \tfrac{1}{2}\left(q_{\rm c}^2 + p_{\rm c}^2\right). \tag{46}$$

The classical Hamiltonian is now $H_{\rm c} = \omega(t)\boldsymbol{I}\cdot\boldsymbol{n}^g(t)$, and the evolution of $q_{\rm c}$ and $p_{\rm c}$ are governed by the canonical equations of motion. It is easy to show that $I = \boldsymbol{I}\cdot\boldsymbol{e}^g(t)$ is an invariant. This leads to a quadratic equation in $q_{\rm c}$ and $p_{\rm c}$:

$$(e_3+e_1)p_{\rm c}^2 + 2e_2q_{\rm c}p_{\rm c} + (e_3-e_1)q_{\rm c}^2 = 2I. \tag{47}$$

If $\boldsymbol{e}^2 = 1$, this describes an ellipse on the $q_{\rm c}p_{\rm c}$ plane, whose area is $2\pi I$. The ellipse changes its shape when $\boldsymbol{e}(t)$ varies with time. If $\boldsymbol{e}(\tau) = \boldsymbol{e}(0)$, the ellipse at the time τ coincides with that at the initial time. This is a classical nonadiabatic cyclic evolution. The q_c and p_c can be expressed in terms of I and its canonical variables θ as

$$q_c(\theta, I, \boldsymbol{e}) = \sqrt{2I(e_3 + e_1)}\cos\theta \qquad p_c(\theta, I, \boldsymbol{e}) = -\sqrt{\frac{2I}{e_3 + e_1}}(e_2\cos\theta + \sin\theta). \tag{48}$$

Using $\boldsymbol{e}^2 = 1$, we have $\mathrm{d}p_c \wedge \mathrm{d}q_c = I\cos^2\theta\,\mathrm{d}e_1 \wedge \mathrm{d}e_2/e_3$, and the contour average is $\langle \mathrm{d}p_c \wedge \mathrm{d}q_c\rangle = I\,\mathrm{d}e_1 \wedge \mathrm{d}e_2/2e_3$. According to [40], $\Delta\theta_g = -\partial_I \int \langle \mathrm{d}p_c \wedge \mathrm{d}q_c\rangle$, we arrive at

$$\Delta\theta_g = -\frac{1}{2}\int_{S_{12}} \frac{\mathrm{d}e_1 \wedge \mathrm{d}e_2}{e_3}. \tag{49}$$

Note that $\mathrm{d}e_1 \wedge \mathrm{d}e_2$ corresponds to $\mathrm{d}S_{12}$ ($-\mathrm{d}S_{12}$) for an anticlockwise (clockwise) trace of $\boldsymbol{e}(t)$, and $e_3 = \sqrt{1 + e_1^2 + e_2^2}$, this is the same as equation (45). It is independent of I.

5. More on cyclic solutions and geometric phases

In the last section we have shown that equation (37) is a sufficient and necessary condition for the existence of cyclic solutions in the time interval $[0, \tau]$. Under this condition, there exists a denumerable set of cyclic solutions. In this section we discuss some special cases where more general cyclic solutions are available. We will see that the simple relation (43) has to be modified.

Before discussing these cases, we define

$$\bar{x}(t) = (\psi(t), x\psi(t)) \qquad \bar{p}(t) = (\psi(t), p\psi(t)) \tag{50}$$

for any state $\psi(t)$, and study their evolution with time. According to the Schrödinger equation, they satisfy the equation of motion:

$$\dot{\bar{x}} = \omega[n_2\bar{x} + (n_1 + n_3)\bar{p}] \qquad \dot{\bar{p}} = \omega[(n_1 - n_3)\bar{x} - n_2\bar{p}]. \tag{51}$$

This is the same as that for the classical variables q_c and p_c, since the Hamiltonian is quadratic in x and p. We denote a two-component column vector $q = (\bar{x}, \bar{p})^{\mathrm{t}}$. Because the above equation is linear, we have

$$q(t) = E_q(t)q(0) \tag{52}$$

where $E_q(t)$ is a 2×2 evolution matrix independent of $q(0)$. Next we define a vector $\boldsymbol{v}(t)$ as

$$v_1 = \tfrac{1}{2}(\bar{x}^2 - \bar{p}^2) \qquad v_2 = -\bar{x}\bar{p} \qquad v_3 = \tfrac{1}{2}(\bar{x}^2 + \bar{p}^2). \tag{53}$$

It is straightforward to show that $\boldsymbol{v}(t)$ satisfies the same equation of motion as $\boldsymbol{e}(t)$ or $\boldsymbol{u}(t)$. Therefore the evolution matrix for $\boldsymbol{v}(t)$ is $E(t)$. On the other hand, the above definition can be written as

$$v_i(t) = \tfrac{1}{2}q^{\mathrm{t}}(t)J_i q(t) \tag{54}$$

where

$$J_1 = \sigma_z \qquad J_2 = -\sigma_x \qquad J_3 = 1 \tag{55}$$

and the σ are Pauli matrices. Substituting equation (52) into equation (54), we have

$$v_i(t) = \tfrac{1}{2}q^{\mathrm{t}}(0)\left[E_q^{\mathrm{t}}(t)J_i E_q(t)\right]q(0). \tag{56}$$

Now any 2×2 matrix can be expanded in terms of the above J_i and $J_0 = \mathrm{i}\sigma_y$. Note that J_i are symmetric while J_0 is antisymmetric, and the matrices $E_q^{\mathrm{t}}(t)J_i E_q(t)$ are symmetric, we have

$$E_q^{\mathrm{t}}(t)J_i E_q(t) = a_{ij}(t)J_j. \tag{57}$$

It is easy to show that $\mathrm{tr}(J_i J_j) = 2\delta_{ij}$, and this yields $a_{ij}(t) = \frac{1}{2}\,\mathrm{tr}\left[E_q^{\mathrm{t}}(t) J_i E_q(t) J_j\right]$. Substituting into equation (56), we obtain $v_i(t) = a_{ij}(t)v_j(0)$. Therefore, $E_{ij}(t) = a_{ij}(t)$, that is

$$E_{ij}(t) = \tfrac{1}{2}\,\mathrm{tr}\left[E_q^{\mathrm{t}}(t) J_i E_q(t) J_j\right]. \tag{58}$$

In other words, if the classical equation of motion is solved, which gives $E_q(t)$, then $E(t)$ can be obtained by simple algebraic calculations. This indicates a relation between our formalism and those of some previous authors, who find the time evolution operator of the Schrödinger equation by solving the classical equation of motion [16, 6]. However, our formalism, where $\boldsymbol{e}(t)$ plays the central role, is more convenient for the discussions of cyclic solutions and geometric phases, for example, in obtaining the necessary and sufficient condition (37). In practical calculations, it is usually more convenient to solve equation (17) directly than using the above relation. However, this relation is convenient for some general discussions. For example, when $E_q(t) = \pm 1$, it is obvious that $E_{ij}(t) = \delta_{ij}$, or $E(t) = 1$. This will be useful below.

Now we go into the main subject of this section. On the premise of equation (39), the time evolution operator is given by equation (40), where $\alpha(\tau)$ depends on the direction of $\boldsymbol{e}_0$. If it happens that $\alpha(\tau)/\pi$ is a rational number, then more cyclic solutions are available. Of special interest are the cases where $\alpha(\tau) = 2N\pi$ and $\alpha(\tau) = (2N+1)\pi$. These will be discussed separately.

5.1. $\alpha(\tau) = 2N\pi$

In this case $U(\tau)$ becomes a c-number. In fact, comparing equations (40) and (4), we find that $U(\tau) = Q(-2\alpha(\tau), \boldsymbol{e}_0)$. If $\alpha(\tau)$ takes the above value, then $U(\tau) = Q(-4N\pi, \boldsymbol{e}_0)$, which has been shown to be a c-number in section 2. The value of this number can be obtained by applying $U(\tau)$ to a specific state, say, $\psi_0^{e_0}$, the ground state of $\boldsymbol{K}\cdot\boldsymbol{e}_0^g$, which gives the result

$$U(\tau) = \mathrm{e}^{\mathrm{i}N\pi}. \tag{59}$$

Several consequences can be deduced in this case.

(1) All solutions are cyclic in the time interval $[0, \tau]$, including nonnormalizable states, though we are only interested in normalizable ones.

(2) Let $\tilde{\boldsymbol{e}}_0 = (\sinh\tilde{\xi}_0\cos\tilde{\phi}_0, \sinh\tilde{\xi}_0\sin\tilde{\phi}_0, \cosh\tilde{\xi}_0)$, where $\tilde{\xi}_0$ and $\tilde{\phi}_0$ are arbitrary. One can choose $\psi(0) = \psi_0^{\tilde{e}_0} = Q(\tilde{\xi}_0, \tilde{\phi}_0)\psi_0$ as an initial state such that $\boldsymbol{u}(0) = \hbar\tilde{\boldsymbol{e}}_0/2$. In the state $\psi(t)$ with the above initial condition $\psi(0)$, we have $\boldsymbol{u}(t) = \hbar\tilde{\boldsymbol{e}}(t)/2$, where $\tilde{\boldsymbol{e}}(t)$ is the solution to equation (17*a*) with the initial condition $\tilde{\boldsymbol{e}}_0$, because $\boldsymbol{u}(t)$ and $\tilde{\boldsymbol{e}}(t)$ satisfy the same equation. Now

$$\boldsymbol{u}(\tau) = (\psi(\tau), \boldsymbol{K}\psi(\tau)) = (\psi(0), U^\dagger(\tau)\boldsymbol{K}U(\tau)\psi(0)) = (\psi(0), \boldsymbol{K}\psi(0)) = \boldsymbol{u}(0). \tag{60}$$

Therefore

$$\tilde{\boldsymbol{e}}(\tau) = \tilde{\boldsymbol{e}}_0. \tag{61}$$

Because $\tilde{\xi}_0$ and $\tilde{\phi}_0$ are arbitrary, this means that the evolution matrix $E(t)$ is a unit matrix at the time τ

$$E(\tau) = 1. \tag{62}$$

In this case, we have obviously $\sigma(\tau) = 1$, that is, all three eigenvalues of $E(\tau)$ are 1. The inverse is not true, however. In some cases we have three eigenvalues all equal to 1, but $E(\tau)$ cannot be diagonalized and is, of course, not a unit matrix (see section 6).

(3) Now we take $\tilde{\boldsymbol{e}}_0$, different from $\boldsymbol{e}_0$, as the initial condition for equation (17*a*), and using $\tilde{\boldsymbol{e}}(t)$ instead of $\boldsymbol{e}(t)$ in equation (34*b*), we obtain

$$U(\tau) = \exp\left[\mathrm{i}\hbar^{-1}\tilde{\alpha}(\tau)\boldsymbol{K}\cdot\tilde{\boldsymbol{e}}_0^g\right] \tag{63}$$

where $\tilde{\alpha}(\tau)$ is given by equation (41), with $\boldsymbol{e}(t), \xi(t)$ and $\phi(t)$ replaced by $\tilde{\boldsymbol{e}}(t)$, $\tilde{\xi}(t)$ and $\tilde{\phi}(t)$, respectively. This must be equal to that in equation (59), however. Thus we should have $\tilde{\alpha}(\tau) = 2\tilde{N}\pi$, and $\tilde{N} - N$ must be an even integer. Actually, we will show that $\tilde{\alpha}(\tau) = \alpha(\tau)$, or $\tilde{N} = N$.

Consider two initial unit vectors $\boldsymbol{e}_0$ and $\tilde{\boldsymbol{e}}_0$, whose difference $\delta\boldsymbol{e}_0 = \tilde{\boldsymbol{e}}_0 - \boldsymbol{e}_0$ is infinitesimal (then $\delta\boldsymbol{e}_0\cdot\boldsymbol{e}_0^g = 0$). The difference in $\alpha(\tau)$, according to equation (41), must be infinitesimal because the difference in $\boldsymbol{e}(t)$, and thus $\xi(t)$ and $\dot{\phi}(t)$ are all infinitesimal. Therefore $\alpha(\tau)$ and thus $\alpha(\tau)/\pi$ are continuous functions of $\boldsymbol{e}_0$. Now that $\alpha(\tau)/\pi$ can take only on even integers, an obvious consequence is that $\alpha(\tau)$ is a constant, independent of ξ_0 and ϕ_0.

(4) Consider a cyclic solution in $[0, \tau]$ with an arbitrary initial condition $\psi(0)$ which is normalizable. The average value of $\boldsymbol{K}$ in $\psi(0)$ is denoted by $\boldsymbol{u}(0)$ as before. Since $\boldsymbol{u}^2(0) \geqslant \hbar^2/4 > 0$, we define $u_0 = \sqrt{\boldsymbol{u}^2(0)}/\hbar$ which is dimensionless, and introduce

$$\boldsymbol{e}_0 = \boldsymbol{u}(0)/\hbar u_0. \tag{64}$$

Obviously, $\boldsymbol{e}_0^2 = 1$, and $e_{03} > 0$ because $u_3(0) > 0$, thus this $\boldsymbol{e}_0$ can be used as the initial condition in equation (17). At later times, $\boldsymbol{e}(t) = \boldsymbol{u}(t)/\hbar u_0$. The dynamic phase is

$$\beta = -\hbar^{-1}\int_0^\tau \omega(t)\boldsymbol{u}(t)\cdot\boldsymbol{n}^g(t)\,\mathrm{d}t. \tag{65}$$

Though $\alpha(\tau) = 2N\pi$ is independent of $\boldsymbol{e}_0$, we must take the one given by equation (64) such that the second term in equation (41) can be related to the dynamic phase above. Then

$$\beta = u_0[\alpha(\tau) + \Delta\theta_{\mathrm{g}}] = u_0(2N\pi + \Delta\theta_{\mathrm{g}}) \tag{66}$$

where $\Delta\theta_{\mathrm{g}}$ is calculated by substituting the above $\boldsymbol{e}(t)$ into equation (44). Because of equation (59), the total phase change is $\delta = N\pi$, mod 2π. Therefore, the geometric phase $\gamma = \delta - \beta$ turns out to be

$$\gamma = -u_0\Delta\theta_{\mathrm{g}} - \left(u_0 - \tfrac{1}{2}\right)2N\pi \qquad \mathrm{mod}\ 2\pi. \tag{67}$$

Here the first term is the familiar one, but an extra term appears, which depends on the initial condition. It vanishes (mod 2π of course) when $u_0 - 1/2$ is an integer, especially when the initial state is an eigenstate of $\boldsymbol{K}\cdot\boldsymbol{e}_0^g$ (it cannot be an eigenstate of $\boldsymbol{K}\cdot\tilde{\boldsymbol{e}}_0^g$ with some other $\tilde{\boldsymbol{e}}_0$ since otherwise $\boldsymbol{u}(0)$ would point in the direction of $\tilde{\boldsymbol{e}}_0$) such that $\boldsymbol{u}(0) = k_n\hbar\boldsymbol{e}_0$ and $u_0 = k_n$. In the latter case it reduces to equation (43) as expected.

(5) We have seen that equation (39) plus the condition $\alpha(\tau) = 2N\pi$ leads to equation (59), and as a result, all solutions are cyclic in the time interval $[0, \tau]$. If $\boldsymbol{e}(t)$ is complicated, however, it is not convenient to use the above criterion because it may be difficult to calculate $\alpha(\tau)$. Thus some other convenient criterion is of interest. Now we give a necessary and sufficient condition for equation (59):

$$E_q(\tau) = 1 \quad\Longleftrightarrow\quad U(\tau) = \mathrm{e}^{\mathrm{i}N\pi}. \tag{68}$$

First, suppose that $U(\tau) = \mathrm{e}^{\mathrm{i}N\pi}$. For arbitrarily given values x_0 and p_0, it is easy to find an initial state $\psi(0)$ such that $\bar{x}(0) = x_0$ and $\bar{p}(0) = p_0$. It is obvious that $\bar{x}(\tau) = \bar{x}(0)$, $\bar{p}(\tau) = \bar{p}(0)$. Since $x(0)$ and $p(0)$ are arbitrary, we obtain $E_q(\tau) = 1$. Second, suppose that $E_q(\tau) = 1$. Then we have $E(\tau) = 1$ as mentioned below equation (58), and $\boldsymbol{e}(\tau) = \boldsymbol{e}_0$ for any $\boldsymbol{e}_0$. Now we choose $\boldsymbol{e}_0 = (0, 0, 1)$, and have from equation (40) $U(\tau) = \exp[\mathrm{i}\hbar^{-1}\alpha(\tau)K_3]$.

On the other hand, $E_q(\tau) = 1$ leads to $\bar{x}(\tau) = \bar{x}(0)$ and $\bar{p}(\tau) = \bar{p}(0)$ for an arbitrary $\psi(0)$, that is

$$\begin{aligned}(\psi(0), U^\dagger(\tau) x U(\tau)\psi(0)) &= (\psi(0), x\psi(0)) \\ (\psi(0), U^\dagger(\tau) p U(\tau)\psi(0)) &= (\psi(0), p\psi(0)).\end{aligned} \tag{69}$$

Since $\psi(0)$ is arbitrary, we should have $U^\dagger(\tau)xU(\tau) = x$ and $U^\dagger(\tau)pU(\tau) = p$. From equation (14), this is valid only when $\alpha(\tau) = 2N\pi$, which leads to $U(\tau) = \mathrm{e}^{\mathrm{i}N\pi}$.

5.2. $\alpha(\tau) = (2N+1)\pi$

In this case, $U(\tau) = \exp\left[\mathrm{i}\hbar^{-1}\alpha(\tau)\boldsymbol{K}\cdot\boldsymbol{e}_0^g\right] = Q(\xi_0,\phi_0)\exp[\mathrm{i}\hbar^{-1}(2N+1)\pi K_3]Q^\dagger(\xi_0,\phi_0)$, where we have used equation (11) in obtaining the second equality. As pointed out in section 2, $\exp[\mathrm{i}\hbar^{-1}(2N+1)\pi K_3]$ commutes with $\boldsymbol{K}$, and thus commutes with $Q(\xi_0,\phi_0)$, we have

$$U(\tau) = \exp\left[\mathrm{i}\hbar^{-1}(2N+1)\pi\boldsymbol{K}\cdot\boldsymbol{e}_0^g\right] = \exp[\mathrm{i}\hbar^{-1}(2N+1)\pi K_3] = \mathrm{e}^{\mathrm{i}N\pi}\exp(\mathrm{i}\hbar^{-1}\pi K_3). \tag{70}$$

Several consequences similar to those in subsection 5.1 can be deduced in this case.

(1) All normalizable solutions with definite parity are cyclic in the time interval $[0,\tau]$. In fact, an initial state with definite parity can be expanded as

$$\psi^+(0) = \sum_{n=0}^{\infty} a_{2n}\psi_{2n} \qquad \psi^-(0) = \sum_{n=0}^{\infty} a_{2n+1}\psi_{2n+1} \tag{71}$$

where the superscript + (−) indicates even (odd) parity, and time evolution does not change the parity of an initial state since $U(t)$ only involves $\boldsymbol{K}$ and $\boldsymbol{K}$ is quadratic in x and p. It is easy to see that

$$\psi^\pm(\tau) = \exp(\mathrm{i}\delta_\pm)\psi^\pm(0) \tag{72}$$

where

$$\delta_\pm = \pm\left(N+\tfrac{1}{2}\right)\pi \qquad \mod 2\pi. \tag{73}$$

(2) Repeat the discussions of the second point in subsection 5.1. Though $U(\tau)$ is not a c-number now, it commutes with $\boldsymbol{K}$. Thus equation (60) is still valid, and so is equation (62).

(3) As before, we take $\tilde{\boldsymbol{e}}_0$, different from $\boldsymbol{e}_0$, as the initial condition for equation (17a), and obtain

$$U(\tau) = \exp\left[\mathrm{i}\hbar^{-1}\tilde{\alpha}(\tau)\boldsymbol{K}\cdot\tilde{\boldsymbol{e}}_0^g\right] = Q(\tilde{\xi}_0,\tilde{\phi}_0)\exp[\mathrm{i}\hbar^{-1}\tilde{\alpha}(\tau)K_3]Q^\dagger(\tilde{\xi}_0,\tilde{\phi}_0). \tag{74}$$

This must be equal to that in equation (70), however. Because $\exp[\mathrm{i}\hbar^{-1}(2N+1)\pi K_3]$ commutes with $Q(\tilde{\xi}_0,\tilde{\phi}_0)$, we have $\exp[\mathrm{i}\hbar^{-1}\tilde{\alpha}(\tau)K_3] = \exp[\mathrm{i}\hbar^{-1}(2N+1)\pi K_3]$. This means that $\tilde{\alpha}(\tau) = (2\tilde{N}+1)\pi$, and $\tilde{N}-N$ must be an even integer. By arguments similar to those in subsection 5.1, we can conclude that $\tilde{\alpha}(\tau) = \alpha(\tau)$.

(4) Consider a cyclic solution in $[0,\tau]$ with a normalizable initial state $\psi(0)$ which is of definite parity. We define $\boldsymbol{e}_0$ as in equation (64) and use it as the initial condition in equation (17), then $\boldsymbol{e}(t) = \boldsymbol{u}(t)/\hbar u_0$. The dynamic phase is of the form in equation (65). As before, we use the above $\boldsymbol{e}(t)$ to calculate $\alpha(\tau)$, then the second term in $\alpha(\tau)$ can be related to the dynamic phase, and

$$\beta = u_0[\alpha(\tau)+\Delta\theta_{\mathrm{g}}] = u_0[(2N+1)\pi+\Delta\theta_{\mathrm{g}}] \tag{75}$$

where $\Delta\theta_{\mathrm{g}}$ is calculated by substituting the above $\boldsymbol{e}(t)$ into equation (44). The total phase change has been given in equation (73). Therefore the geometric phase turns out to be

$$\gamma_\pm = -u_0\Delta\theta_{\mathrm{g}} - \left(u_0 \mp \tfrac{1}{2}\right)(2N+1)\pi \qquad \mod 2\pi. \tag{76}$$

Here an extra term appears once again, which depends on the initial condition. It vanishes (mod 2π of course) when $u_0 \mp 1/2$ happens to be an even integer. For example, if the initial state is an eigenstate of $\boldsymbol{K} \cdot \boldsymbol{e}_0^g$ (then it is of definite parity) such that $\boldsymbol{u}(0) = k_n \hbar \boldsymbol{e}_0$ and $u_0 = k_n$, then the extra term vanishes as expected.

(5) As in subsection 5.1, we give a necessary and sufficient condition for equation (70) which may be more convenient:

$$E_q(\tau) = -1 \quad \Longleftrightarrow \quad U(\tau) = \exp[\mathrm{i}\hbar^{-1}(2N+1)\pi K_3]. \tag{77}$$

First, suppose that $U(\tau) = \exp[\mathrm{i}\hbar^{-1}(2N+1)\pi K_3]$. Using equation (14), it is easy to show that

$$U^\dagger(\tau) x U(\tau) = -x \qquad U^\dagger(\tau) p U(\tau) = -p. \tag{78}$$

For arbitrarily given values x_0 and p_0, it is easy to find an initial state $\psi(0)$ such that $\bar{x}(0) = x_0$ and $\bar{p}(0) = p_0$. The above equation yields $\bar{x}(\tau) = -\bar{x}(0)$, $\bar{p}(\tau) = -\bar{p}(0)$. Since $x(0)$ and $p(0)$ are arbitrary, we obtain $E_q(\tau) = -1$. Second, suppose that $E_q(\tau) = -1$. Then we have $E(\tau) = 1$, and $\boldsymbol{e}(\tau) = \boldsymbol{e}_0$ for any $\boldsymbol{e}_0$. Now we choose $\boldsymbol{e}_0 = (0, 0, 1)$, and have from equation (40) $U(\tau) = \exp[\mathrm{i}\hbar^{-1}\alpha(\tau)K_3]$. On the other hand, $E_q(\tau) = -1$ leads to $\bar{x}(\tau) = -\bar{x}(0)$ and $\bar{p}(\tau) = -\bar{p}(0)$ for an arbitrary $\psi(0)$. Thus equation (78) must be true, and from equation (14), we have $\alpha(\tau) = (2N+1)\pi$, which leads to $U(\tau) = \exp[\mathrm{i}\hbar^{-1}(2N+1)\pi K_3]$.

5.3. Other cases

Finally we briefly discuss the case where $\alpha(\tau)/\pi$ is a rational number other than an integer. More specifically, let $\alpha(\tau) = r_0\pi/s_0$, where both r_0 and s_0 are integers and prime to each other, and $s_0 \geqslant 2$. In addition to the denumerable set of cyclic solutions discussed before, we also have other ones in this case. For example, the initial conditions

$$\psi(0) = \sum_{s=0}^{\infty} a_s \psi_{2s_0 s}^{e_0} = Q(\xi_0, \phi_0) \sum_{s=0}^{\infty} a_s \psi_{2s_0 s} \qquad \sum_{s=0}^{\infty} |a_s|^2 = 1 \tag{79}$$

give cyclic solutions in the time interval $[0, \tau]$. In fact, it is easy to show that $\psi(\tau) = \mathrm{e}^{\mathrm{i}\delta}\psi(0)$ with $\delta = \alpha(\tau)/2 = r_0\pi/2s_0$. In the above initial states, we have $\boldsymbol{u}(0) = \hbar u_0 \boldsymbol{e}_0$ where $u_0 = 2s_0\bar{s} + 1/2$ and $\bar{s} = \sum_{s=0}^{\infty} s|a_s|^2$. The dynamic phase is $\beta = u_0[\alpha(\tau) + \Delta\theta_{\mathrm{g}}]$. Thus the geometric phase is

$$\gamma = -u_0\Delta\theta_{\mathrm{g}} - 2r_0\bar{s}\pi \qquad \mathrm{mod}\ 2\pi. \tag{80}$$

We find once again that an extra term appears in the geometric phase. This term disappears when $\bar{s}$ happens to be an integer, especially when the above initial state involves only one term.

6. Some examples with explicit results

In this section we calculate some simple examples where explicit analytic results are available. The Hamiltonian involved in these examples may be elliptic, hyperbolic or critical. The motion of the particle exhibits various possible patterns. A normalizable state can be regarded as a wave packet in the configuration space. The centre of the wave packet is characterized by $\bar{x}$ and its velocity characterized by $\bar{p}$, and their evolution is governed by the matrix $E_q(t)$. If $E_q(t)$ is finite at all times (for example, its elements are trigonometric functions of t), then the motion is said to be finite, because both $\bar{x}$ and $\bar{p}$ will remain finite at all times. Otherwise it is said to be infinite. The latter case contains still different patterns, for example, $E_q(t)$ may increase with time or may be oscillating with increasing amplitude. Other quantities that characterize a wave packet are mainly the variances $\Delta x = \sqrt{\langle (x-\bar{x})^2\rangle}$ and $\Delta p = \sqrt{\langle (p-\bar{p})^2\rangle}$. They can be regarded as the width of the wave packet in the coordinate and momentum space, respectively. From equation (58) we see that if $E_q(t)$ is finite (infinite), then $E(t)$ is essentially finite (infinite) as well. We will see that the time evolution of $(\Delta x)^2$ and $(\Delta p)^2$ is essentially governed by $E(t)$, thus the width and the position of a wave packet exhibit similar patterns of motion in this system. In other words, if the wave packet moves in a confined region, then its width also remains finite, and the opposite is also true. In fact, we can define a new vector $\boldsymbol{w}(t)$ by

$$\boldsymbol{w}(t) = \boldsymbol{u}(t) - \boldsymbol{v}(t). \tag{81}$$

Since both $\boldsymbol{u}(t)$ and $\boldsymbol{v}(t)$ satisfy the same equation as $\boldsymbol{e}(t)$, it is obvious that $\boldsymbol{w}(t)$ also satisfies the same equation, and thus its time evolution is governed by the matrix $E(t)$. On the other hand, it is easy to show that

$$w_1 = \tfrac{1}{2}[(\Delta x)^2 - (\Delta p)^2] \qquad w_3 = \tfrac{1}{2}[(\Delta x)^2 + (\Delta p)^2] \tag{82a}$$

$$w_2 = -\tfrac{1}{2}\langle (x-\bar{x})(p-\bar{p}) + (p-\bar{p})(x-\bar{x})\rangle. \tag{82b}$$

Therefore,

$$(\Delta x)^2 = w_3 + w_1 \qquad (\Delta p)^2 = w_3 - w_1 \tag{83}$$

and their time evolution is essentially governed by $E(t)$.

We will see that the pattern of motion does not have a simple relation with the nature of the Hamiltonian. For a Hamiltonian with a definite nature, say, elliptic, the wave packet may exhibit different patterns of motion if some parameter in the Hamiltonian takes different values. And, when the parameter goes through some critical value, the motion changes from one pattern to another. This is somewhat like a phase transition, and seems not to have been noticed before.

6.1. $\boldsymbol{n}$ = constant

The first case is the simplest one where $\boldsymbol{n}$ is a constant vector and $\omega(t)$ is an arbitrary function of time. Let

$$\varphi(t) = \int_0^t \omega(t')\,\mathrm{d}t' \tag{84}$$

and if $\boldsymbol{e}(t) = \boldsymbol{e}[\varphi(t)]$ obeys

$$\boldsymbol{e}'(\varphi) = -2\boldsymbol{n}^g \times \boldsymbol{e}^g(\varphi) \tag{85}$$

where the prime indicates a derivative with respect to φ, then equation (17a) is satisfied. This is similar to equation (5) and can be solved in the same way. The result depends on the nature of $\boldsymbol{n}$, and will be given separately.

(1) If $\boldsymbol{n}^2 = 1$, the solution reads

$$\boldsymbol{e}(t) = [\boldsymbol{e}_0 - (\boldsymbol{e}_0\cdot\boldsymbol{n}^g)\boldsymbol{n}]\cos 2\varphi + \boldsymbol{e}_0^g \times \boldsymbol{n}^g \sin 2\varphi + (\boldsymbol{e}_0\cdot\boldsymbol{n}^g)\boldsymbol{n}. \tag{86}$$

To obtain the evolution matrix $E(t)$, we should write down the components of $\boldsymbol{n}$:

$$\boldsymbol{n} = (\sinh\xi_n \cos 2\varphi_n, \sinh\xi_n \sin 2\varphi_n, \cosh\xi_n) \tag{87}$$

where ξ_n and φ_n are arbitrary. The matrix $E(t)$ is somewhat complicated, but it can be written as the product of several simple matrices

$$E(t) = R(\varphi_n)S(\xi_n, 1)W_+^{(a)}(\varphi)S^{-1}(\xi_n, 1)R^{-1}(\varphi_n) \tag{88}$$

where

$$W_+^{(a)}(\varphi) = \begin{pmatrix} \cos 2\varphi & -\sin 2\varphi & 0 \\ \sin 2\varphi & \cos 2\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{89}$$

and the matrices R and S, which will be repeatedly used below, are defined by

$$R(\varphi_n) = \begin{pmatrix} \cos 2\varphi_n & -\sin 2\varphi_n & 0 \\ \sin 2\varphi_n & \cos 2\varphi_n & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad S(\xi_n, \epsilon) = \begin{pmatrix} \epsilon \cosh \xi_n & 0 & \sinh \xi_n \\ 0 & 1 & 0 \\ \sinh \xi_n & 0 & \epsilon \cosh \xi_n \end{pmatrix} \tag{90}$$

where $\epsilon = \pm 1$. The eigenvalues of $E(t)$ can be easily found to be $\{1, \mathrm{e}^{\mathrm{i}2\varphi}, \mathrm{e}^{-\mathrm{i}2\varphi}\}$. The eigenvector corresponding to the eigenvalue 1 is $\boldsymbol{\eta} = \boldsymbol{n}$. This can be easily seen from equation (86), where $\boldsymbol{e}_0 = \boldsymbol{n}$ leads to $\boldsymbol{e}(t) = \boldsymbol{n}$ at any time t. Because $\boldsymbol{n}^2 = 1$, there exist normalizable cyclic solutions at any time interval $[0, \tau]$. However, most of these cyclic solutions are trivial ones. Actually, the time evolution operator can be obtained by direct integration in this case:

$$U(t) = \exp(-\mathrm{i}\hbar^{-1}\varphi \boldsymbol{K} \cdot \boldsymbol{n}^g). \tag{91}$$

If $\psi(0)$ is an eigenstate of $\boldsymbol{K} \cdot \boldsymbol{n}^g$, then $\psi(t)$ is different from $\psi(0)$ only by a phase factor, thus it is cyclic in any time interval but is trivial. Only when $\varphi(\tau)/\pi$ takes rational numbers do we have nontrivial cyclic solutions in $[0, \tau]$.

The evolution matrix $E_q(t)$ can be similarly found to be

$$E_q(t) = R_q(\varphi_n) W_{q+}^{(a)}(\varphi) R_q^{-1}(\varphi_n) \tag{92}$$

where

$$W_{q+}^{(a)}(\varphi) = \begin{pmatrix} \cos\varphi & \exp(\xi_n)\sin\varphi \\ -\exp(-\xi_n)\sin\varphi & \cos\varphi \end{pmatrix} \tag{93}$$

and

$$R_q(\varphi_n) = \begin{pmatrix} \cos\varphi_n & \sin\varphi_n \\ -\sin\varphi_n & \cos\varphi_n \end{pmatrix}. \tag{94}$$

Obviously, when $\varphi(\tau) = -N\pi$, $E_q(\tau) = (-1)^N$ and $E(\tau) = 1$. This includes the two most interesting cases discussed in section 5.

The time evolution operator (91) can also be obtained by our method. Taking the simplest solution $\boldsymbol{e}(t) = \boldsymbol{n}$, we have $\xi(t) = \xi_0 = \xi_n$, $\phi(t) = \phi_0 = 2\varphi_n$ and $\alpha(t) = -\varphi(t)$. When these are substituted into equation (34*b*), we obtain the above result. Moreover, the above condition $\varphi(\tau) = -N\pi$ for $E_q(\tau) = (-1)^N$ is equivalent to $\alpha(\tau) = N\pi$, as expected.

To verify equation (67) by explicit calculations, we take $\boldsymbol{n} = (0, 0, 1)$, then $U(t) = \exp[-\mathrm{i}\hbar^{-1}\varphi(t)K_3]$. When $\varphi(\tau) = -2N\pi$, we have $U(\tau) = \mathrm{e}^{\mathrm{i}N\pi}$. In the time interval $[0, \tau]$, all solutions are cyclic with $\delta = N\pi$, mod 2π. Let the initial state have $\boldsymbol{u}(0) = \hbar u_0 \boldsymbol{e}_0$, where $\boldsymbol{e}_0 = (\sinh\xi_0 \cos\phi_0, \sinh\xi_0 \sin\phi_0, \cosh\xi_0)$. Then $\boldsymbol{u}(t) = \hbar u_0 \boldsymbol{e}(t)$, where $\boldsymbol{e}(t) = (\sinh\xi_0 \cos(2\varphi + \phi_0), \sinh\xi_0 \sin(2\varphi + \phi_0), \cosh\xi_0)$. It is then easy to find that $\beta = 2N\pi u_0 \cosh\xi_0$, and $\gamma = N\pi(1 - 2u_0 \cosh\xi_0)$. On the other hand, the Hannay angle can be found to be $\Delta\theta_{\mathrm{g}} = 2N\pi(\cosh\xi_0 - 1)$. With these results it is easy to see that equation (67) is valid.

(2) If $\boldsymbol{n}^2 = -1$, the solution reads

$$\boldsymbol{e}(t) = [\boldsymbol{e}_0 + (\boldsymbol{e}_0 \cdot \boldsymbol{n}^g)\boldsymbol{n}] \cosh 2\varphi + \boldsymbol{e}_0^g \times \boldsymbol{n}^g \sinh 2\varphi - (\boldsymbol{e}_0 \cdot \boldsymbol{n}^g)\boldsymbol{n}. \tag{95}$$

To obtain the evolution matrix $E(t)$, we should write down the components of $\boldsymbol{n}$:

$$\boldsymbol{n} = (\cosh\xi_n \cos 2\varphi_n, \cosh\xi_n \sin 2\varphi_n, \sinh\xi_n) \tag{96}$$

where ξ_n and φ_n are arbitrary. The matrix $E(t)$ is given by

$$E(t) = R(\varphi_n)S(\xi_n, 1)W_-^{(a)}(\varphi)S^{-1}(\xi_n, 1)R^{-1}(\varphi_n) \tag{97}$$

where

$$W_-^{(a)}(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh 2\varphi & -\sinh 2\varphi \\ 0 & -\sinh 2\varphi & \cosh 2\varphi \end{pmatrix}. \tag{98}$$

The eigenvalues of $E(t)$ can be easily found to be $\{1, \mathrm{e}^{2\varphi}, \mathrm{e}^{-2\varphi}\}$. The eigenvector corresponding to the eigenvalue 1 is $\boldsymbol{\eta} = \boldsymbol{n}$. Because $\boldsymbol{n}^2 = -1$, there exists no normalizable cyclic solution at any time interval $[0, \tau]$. The evolution matrix $E_q(t)$ can be found to be

$$E_q(t) = R_q(\varphi_n)W_{q-}^{(a)}(\varphi)R_q^{-1}(\varphi_n) \tag{99}$$

where

$$W_{q-}^{(a)}(\varphi) = \begin{pmatrix} \cosh\varphi & \exp(\xi_n)\sinh\varphi \\ \exp(-\xi_n)\sinh\varphi & \cosh\varphi \end{pmatrix}. \tag{100}$$

In this case $E_q(\tau)$ cannot take the value ± 1, otherwise one should obtain $W_{q-}^{(a)}(\varphi) = \pm 1$. The latter is obviously impossible, unless $\varphi(\tau)$ returns to 0, but this is not of interest.

(3) If $\boldsymbol{n}^2 = 0$, the solution reads

$$\boldsymbol{e}(t) = 2\varphi^2(\boldsymbol{e}_0 \cdot \boldsymbol{n}^g)\boldsymbol{n} + 2\varphi\boldsymbol{e}_0^g \times \boldsymbol{n}^g + \boldsymbol{e}_0. \tag{101}$$

Up to a constant factor, which may be absorbed in $\omega(t)$, the components of $\boldsymbol{n}$ may be written as

$$\boldsymbol{n} = (\cos 2\varphi_n, \sin 2\varphi_n, 1) \tag{102}$$

where φ_n is arbitrary. The matrix $E(t)$ is given by

$$E(t) = R(\varphi_n)W_0^{(a)}(\varphi)R^{-1}(\varphi_n) \tag{103}$$

where

$$W_0^{(a)}(\varphi) = \begin{pmatrix} 1-2\varphi^2 & -2\varphi & 2\varphi^2 \\ 2\varphi & 1 & -2\varphi \\ -2\varphi^2 & -2\varphi & 1+2\varphi^2 \end{pmatrix}. \tag{104}$$

The eigenvalues of $E(t)$ can be easily found to be $\{1, 1, 1\}$, but there is only one eigenvector $\boldsymbol{\eta} = \boldsymbol{n}$ if $\varphi \neq 0$. Thus we find an example where all eigenvalues are 1 but $E(t)$ cannot be diagonalized, let alone be a unit matrix. Because $\boldsymbol{n}^2 = 0$, there exists no normalizable cyclic solution at any time interval $[0, \tau]$. The evolution matrix $E_q(t)$ can be found to be

$$E_q(t) = R_q(\varphi_n)W_{q0}^{(a)}(\varphi)R_q^{-1}(\varphi_n) \tag{105}$$

where

$$W_{q0}^{(a)}(\varphi) = \begin{pmatrix} 1 & 2\varphi \\ 0 & 1 \end{pmatrix}. \tag{106}$$

In this case $E_q(\tau)$ cannot take the value ± 1 either, unless $\varphi(\tau)$ returns to 0 but this is not of interest.

From the above results we see that the motion is finite when the Hamiltonian is elliptic and infinite (assume that $\varphi(t)$ can reach arbitrarily large values, for example, $\varphi(t)$ is proportional to t) in the other cases. But note that $E(t)$ and $E_q(t)$ are polynomials of φ in the critical case and are exponential functions of it in the hyperbolic case. If $\boldsymbol{n}$ is time dependent, however, the situation is not so simple. We will see that for a Hamiltonian of a definite nature, all patterns of motion are possible.

6.2. $\boldsymbol{n} = (n_1 \cos 2\varphi, n_1 \sin 2\varphi, n_3)$

Here n_1 and n_3 are constants, satisfying $\boldsymbol{n}^2 = n_3^2 - n_1^2 = \pm 1$ or 0, and $\varphi = \varphi(t)$ is an arbitrary function of t with $\varphi(0) = 0$. For convenience, we assume henceforth that $\varphi(t)$ increases with t monotonically and goes to infinity when t does. If $\omega(t) = \lambda\dot{\varphi}(t)$ where λ is a constant, analytic solutions are available.

By making a time-dependent unitary transformation $U(t) = \exp[-\mathrm{i}\hbar^{-1}\varphi K_3]\tilde{U}(t)$, it is not difficult to obtain

$$U(t) = \exp(-\mathrm{i}\hbar^{-1}\varphi K_3) \exp\left(-\mathrm{i}\hbar^{-1}\Lambda\varphi \boldsymbol{K} \cdot \boldsymbol{n}_f^g\right) \tag{107}$$

where $\Lambda > 0$ and $\boldsymbol{n}_f^2 = \pm 1$ or 0, and

$$\Lambda \boldsymbol{n}_f = (\lambda n_1, 0, \lambda n_3 - 1). \tag{108}$$

If $\boldsymbol{n}_f^2 = 1$, we see that cyclic solutions are available in $[0, \tau]$ if the initial state is an eigenstate of $\boldsymbol{K} \cdot \boldsymbol{n}_f^g$ and $\varphi(\tau) = N\pi$, and more general cyclic solutions are available in some time interval is Λ happens to be a rational number. However, for other time intervals, or when $\boldsymbol{n}_f^2 = -1$ or 0, the above time evolution operator, though explicit, does not tell much about cyclic solutions. Consequently, it is necessary to study the matrices $E(t)$ and $E_q(t)$.

If $\boldsymbol{e}(t) = \boldsymbol{e}[\varphi(t)]$ obeys

$$\boldsymbol{e}'(\varphi) = -2\lambda \boldsymbol{n}^g(\varphi) \times \boldsymbol{e}^g(\varphi) \tag{109}$$

then equation (17*a*) is satisfied. Now we make a time-dependent linear transformation, written in column vector form:

$$e(\varphi) = R(\varphi) f(\varphi) \tag{110}$$

where the matrix R is defined in equation (90), but now the independent variable φ is time dependent. The reduced equation for $f(\varphi)$, written in ordinary vector form, reads

$$\boldsymbol{f}'(\varphi) = -2\Lambda \boldsymbol{n}_f^g \times \boldsymbol{f}^g(\varphi). \tag{111}$$

Now $\boldsymbol{n}_f$ is a constant vector, so the reduced equation is easy to solve. The form of the solution depends on the nature of $\boldsymbol{n}_f$, and several cases should be treated separately.

(1) If $\boldsymbol{n}^2 = 1$ and $\lambda \in (-\infty, n_3 - |n_1|) \cup (n_3 + |n_1|, +\infty)$, or $\boldsymbol{n}^2 = -1$ and $\lambda \in (-n_3 - |n_1|, -n_3 + |n_1|)$, or $\boldsymbol{n}^2 = 0$ and $\lambda n_3 < 1/2$, then $(\lambda n_3 - 1)^2 > (\lambda n_1)^2$. In this case we define

$$\Lambda = \sqrt{(\lambda n_3 - 1)^2 - (\lambda n_1)^2} \qquad \cosh\xi_n = |\lambda n_3 - 1|/\Lambda \qquad \sinh\xi_n = \lambda n_1/\Lambda \tag{112}$$

then

$$\boldsymbol{n}_f = (\sinh\xi_n, 0, \epsilon\cosh\xi_n) \tag{113}$$

where $\epsilon = \epsilon(\lambda n_3 - 1)$ is the sign function of $\lambda n_3 - 1$, that is, it equals 1 (-1) if $\lambda n_3 - 1 > 0$ (<0). The solution is

$$E(t) = R(\varphi) S(\xi_n, \epsilon) W_+^{(b)}(\varphi) S^{-1}(\xi_n, \epsilon) \qquad E_q(t) = R_q(\varphi) W_{q+}^{(b)}(\varphi) \tag{114}$$

where

$$W_+^{(b)}(\varphi) = \begin{pmatrix} \cos 2\Lambda\varphi & -\sin 2\Lambda\varphi & 0 \\ \sin 2\Lambda\varphi & \cos 2\Lambda\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$W_{q+}^{(b)}(\varphi) = \begin{pmatrix} \cos\Lambda\varphi & \epsilon\exp(\epsilon\xi_n)\sin\Lambda\varphi \\ -\epsilon\exp(-\epsilon\xi_n)\sin\Lambda\varphi & \cos\Lambda\varphi \end{pmatrix} \tag{115}$$

and the matrix R_q is defined in equation (94), but now the independent variable φ is time dependent. In this case the motion is finite, no matter whether the Hamiltonian is elliptic, hyperbolic or critical, as long as λ belongs to the appropriate region.

The eigenvector of $E(t)$ corresponding to the eigenvalue 1 is (not normalized)

$$\boldsymbol{\eta}(t) = (\epsilon \sinh \xi_n \sin \Lambda\varphi \cos \varphi, \epsilon \sinh \xi_n \sin \Lambda\varphi \sin \varphi, \cosh \xi_n \sin \Lambda\varphi \cos \varphi + \epsilon \cos \Lambda\varphi \sin \varphi). \tag{116}$$

If $\sin \varphi = 0$ but $\sin \Lambda\phi \neq 0$ at time τ, or $\sin \Lambda\varphi = 0$ but $\sin \phi \neq 0$ (if both are zero, then $E(\tau) = 1$), then $\boldsymbol{\eta}^2(\tau) > 0$ and there are cyclic solutions in the time interval $[0, \tau]$. In the general case, we have

$$\boldsymbol{\eta}^2(t) = (\sinh^2 \xi_n \sin^2 \Lambda\varphi + 1) \sin^2[\varphi + \epsilon \arctan(\cosh \xi_n \tan \Lambda\varphi)] - \sinh^2 \xi_n \sin^2 \Lambda\varphi. \tag{117}$$

This may be either positive or negative. If at time τ the quantity in the square bracket is close to $(N + 1/2)\pi$, then $\boldsymbol{\eta}^2(\tau) > 0$, and there are cyclic solutions in the time interval $[0, \tau]$. If the quantity is close to $N\pi$, and $\sin \Lambda\varphi$ is not too small, then $\boldsymbol{\eta}^2(\tau) < 0$, and there is no cyclic solution in the time interval $[0, \tau]$.

If Λ is a rational number, then there exist some τ such that $E_q(\tau) = \pm 1$ and $E(\tau) = 1$. This is the case where more general cyclic solutions are available.

The time evolution operator can be easily obtained by our formalism in this case. We take the simple solution $\boldsymbol{e}(t) = (\epsilon \sinh \xi_n \cos 2\varphi, \epsilon \sinh \xi_n \sin 2\varphi, \cosh \xi_n)$ and using equation (34*a*), after some operator algebra, we arrive at the result (107). For the subsequent cases, however, our formalism is not convenient in calculating the time evolution operator, since it is not easy to find a solution that is sufficiently simple and satisfies equation (17*c*). Instead, the time-dependent unitary transformation that leads to the result (107) is convenient in this respect. However, as pointed out before, the result (107) does not tell much about cyclic solutions.

(2) If $\boldsymbol{n}^2 = 1$ and $\lambda \in (n_3 - |n_1|, n_3 + |n_1|)$, or $\boldsymbol{n}^2 = -1$ and $\lambda \in (-\infty, -n_3 - |n_1|) \cup (-n_3 + |n_1|, +\infty)$, or $\boldsymbol{n}^2 = 0$ and $\lambda n_3 > 1/2$, then $(\lambda n_3 - 1)^2 < (\lambda n_1)^2$. In this case we define

$$\Lambda = \sqrt{(\lambda n_1)^2 - (\lambda n_3 - 1)^2} \qquad \cosh \xi_n = |\lambda n_1|/\Lambda \qquad \sinh \xi_n = (\lambda n_1 - 1)/\Lambda \tag{118}$$

then

$$\boldsymbol{n}_f = (\epsilon \cosh \xi_n, 0, \sinh \xi_n) \tag{119}$$

where $\epsilon = \epsilon(\lambda n_1)$. The solution is

$$E(t) = R(\varphi) S(\xi_n, \epsilon) W_-^{(b)}(\varphi) S^{-1}(\xi_n, \epsilon) \qquad E_q(t) = R_q(\varphi) W_{q-}^{(b)}(\varphi) \tag{120}$$

where

$$W_-^{(b)}(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh 2\Lambda\varphi & -\sinh 2\Lambda\varphi \\ 0 & -\sinh 2\Lambda\varphi & \cosh 2\Lambda\varphi \end{pmatrix}$$
$$W_{q-}^{(b)}(\varphi) = \begin{pmatrix} \cosh \Lambda\varphi & \epsilon \exp(\epsilon\xi_n) \sinh \Lambda\varphi \\ \epsilon \exp(-\epsilon\xi_n) \sinh \Lambda\varphi & \cosh \Lambda\varphi \end{pmatrix}. \tag{121}$$

In this case the motion is oscillating with exponentially increasing amplitude, no matter whether the Hamiltonian is elliptic, hyperbolic or critical, as long as λ belongs to the appropriate region.

The eigenvector of $E(t)$ corresponding to the eigenvalue 1 is (not normalized)

$$\boldsymbol{\eta}(t) = (\epsilon \cosh \xi_n \sinh \Lambda\varphi \cos \varphi, \epsilon \cosh \xi_n \sinh \Lambda\varphi \sin \varphi, \sinh \xi_n \sinh \Lambda\varphi \cos \varphi + \cosh \Lambda\varphi \sin \varphi). \tag{122}$$

If $\sin\varphi = 0$ but $\varphi \neq 0$ at time τ, then $\boldsymbol{\eta}^2(\tau) < 0$ and there is no cyclic solution in the time interval $[0, \tau]$. In the general case, we have

$$\boldsymbol{\eta}^2(t) = (\cosh^2\xi_n \sinh^2\Lambda\varphi + 1)\sin^2[\varphi + \arctan(\sinh\xi_n \tanh\Lambda\varphi)] - \cosh^2\xi_n \sinh^2\Lambda\varphi. \tag{123}$$

This may be either positive or negative. If at time τ the quantity in the square bracket is close to $(N + 1/2)\pi$, then $\boldsymbol{\eta}^2(\tau) > 0$, and there are cyclic solutions in the time interval $[0, \tau]$. If the quantity is close to $N\pi$, and φ is not too small, then $\boldsymbol{\eta}^2(\tau) < 0$, and there is no cyclic solution in the time interval $[0, \tau]$. In the present case there exists no τ such that $\varphi(\tau) \neq 0$ and $E_q(\tau) = \pm 1$.

(3) If $\boldsymbol{n}^2 = 1$ and $\lambda = n_3 \pm |n_1|$, or $\boldsymbol{n}^2 = -1$ and $\lambda = -n_3 \pm |n_1|$, or $\boldsymbol{n}^2 = 0$ and $\lambda n_3 = 1/2$, then $(\lambda n_3 - 1)^2 = (\lambda n_1)^2$. Let $\epsilon = (\lambda n_3 - 1)/\lambda n_1 = \pm 1$. The solution is

$$E(t) = R(\varphi)D^{(b)}(\varphi) \qquad E_q(t) = R_q(\varphi)W_{q0}^{(b)}(\varphi) \tag{124}$$

where

$$D^{(b)}(\varphi) = \begin{pmatrix} 1 - 2(\lambda n_1)^2\varphi^2 & -2\epsilon\lambda n_1\varphi & 2\epsilon(\lambda n_1)^2\varphi^2 \\ 2\epsilon\lambda n_1\varphi & 1 & -2\lambda n_1\varphi \\ -2\epsilon(\lambda n_1)^2\varphi^2 & -2\lambda n_1\varphi & 1 + 2(\lambda n_1)^2\varphi^2 \end{pmatrix}$$
$$W_{q0}^{(b)}(\varphi) = \begin{pmatrix} 1 & (1+\epsilon)\lambda n_1\varphi \\ (1-\epsilon)\lambda n_1\varphi & 1 \end{pmatrix}. \tag{125}$$

In this case the motion is oscillating with polynomially increasing amplitude, no matter the Hamiltonian is elliptic, hyperbolic or critical, as long as λ takes the appropriate value.

The eigenvector of $E(t)$ corresponding to the eigenvalue 1 is (not normalized)

$$\boldsymbol{\eta}(t) = (\lambda n_1\varphi\cos\varphi, \lambda n_1\varphi\sin\varphi, \epsilon\lambda n_1\varphi\cos\varphi + \sin\varphi). \tag{126}$$

If $\sin\varphi = 0$ at time τ, then $\boldsymbol{\eta}^2(\tau) = 0$ and there is no cyclic solution in the time interval $[0, \tau]$. In the general case, we have

$$\boldsymbol{\eta}^2(t) = \left(\lambda^2 n_1^2\varphi^2 + 1\right)\sin^2[\varphi + \epsilon\arctan(\lambda n_1\varphi)] - \lambda^2 n_1^2\varphi^2. \tag{127}$$

This may be either positive or negative. If at time τ the quantity in the square bracket is close to $(N + 1/2)\pi$, then $\boldsymbol{\eta}^2(\tau) > 0$, and there are cyclic solutions in the time interval $[0, \tau]$. If the quantity is close to $N\pi$, and φ is not too small, then $\boldsymbol{\eta}^2(\tau) < 0$, and there is no cyclic solution in the time interval $[0, \tau]$. In the present case there exists no τ such that $\varphi(\tau) \neq 0$ and $E_q(\tau) = \pm 1$.

It should be noted that all three eigenvalues of $E(t)$ are 1 when $\varphi(t) = N\pi$, and both of the eigenvalues of $E_q(t)$ are 1 when $\varphi(t) = 2N\pi$, but they are not unit matrices unless $N = 0$.

From the above results we see that for a Hamiltonian of a definite nature, say, elliptic, the motion may have all possible patterns, depending on the value of λ. When $\lambda < n_3 - |n_1|$ or $\lambda > n_3 + |n_1|$ the motion is finite. When $n_3 - |n_1| < \lambda < n_3 + |n_1|$, the motion is oscillating with exponentially increasing amplitude. At the two critical values $\lambda = n_3 \pm |n_1|$ the motion is oscillating with polynomially increasing amplitude. Something like a phase transition happens here. When λ goes through the critical values, the motion changes from one pattern to another, and at the critical values the motion exhibits an independent pattern. A similar situation will be seen in the next subsection.

A common feature of the three different cases above is that in any time interval $(\tau_0, +\infty)$ with $\tau_0 > 0$ one can always find some τ such that cyclic solutions are available in $[0, \tau]$.

6.3. $\boldsymbol{n} = (n_1, n_3 \sinh 2\varphi, n_3 \cosh 2\varphi)$

As before, n_1 and n_3 are constants, satisfying $\boldsymbol{n}^2 = n_3^2 - n_1^2 = \pm 1$ or 0, and $\varphi = \varphi(t)$ has the same property as in subsection 6.2. If $\omega(t) = \lambda\dot{\varphi}(t)$ where λ is a constant, analytic solutions are available. The time evolution operator can be obtained in a way similar to that in subsection 6.2. The result is

$$U(t) = \exp(-\mathrm{i}\hbar^{-1}\varphi K_1) \exp\left(-\mathrm{i}\hbar^{-1}\Lambda\varphi \boldsymbol{K} \cdot \boldsymbol{n}_f^g\right) \tag{128}$$

where $\Lambda > 0$ and $\boldsymbol{n}_f^2 = \pm 1$ or 0, and

$$\Lambda \boldsymbol{n}_f = (\lambda n_1 + 1, 0, \lambda n_3). \tag{129}$$

Compared with the case in subsection 6.2, here even less about cyclic solutions can be seen from the above result. Consequently, it is necessary to study the matrices $E(t)$ and $E_q(t)$.

If $\boldsymbol{e}(t) = \boldsymbol{e}[\varphi(t)]$ obeys an equation of the form (109), then equation (17*a*) is satisfied. As before we make a time-dependent linear transformation, written in column vector form:

$$e(\varphi) = T(\varphi) f(\varphi) \tag{130}$$

where the matrix T, and another matrix T_q that will be used below, are defined as

$$T(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh 2\varphi & \sinh 2\varphi \\ 0 & \sinh 2\varphi & \cosh 2\varphi \end{pmatrix} \qquad T_q(\varphi) = \begin{pmatrix} \cosh\varphi & -\sinh\varphi \\ -\sinh\varphi & \cosh\varphi \end{pmatrix}. \tag{131}$$

The reduced equation for $f(\varphi)$, written in ordinary vector form, has the form of equation (111), but where $\Lambda \boldsymbol{n}_f$ is given by equation (129). As before, several cases are to be treated separately.

(1) If $\boldsymbol{n}^2 = 1$ and $\lambda \in (-\infty, n_1 - |n_3|) \cup (n_1 + |n_3|, +\infty)$, or $\boldsymbol{n}^2 = -1$ and $\lambda \in (-n_1 - |n_3|, -n_1 + |n_3|)$, or $\boldsymbol{n}^2 = 0$ and $\lambda n_1 < -1/2$, then $(\lambda n_3)^2 > (\lambda n_1 + 1)^2$. In this case we define

$$\Lambda = \sqrt{(\lambda n_3)^2 - (\lambda n_1 + 1)^2} \qquad \cosh\xi_n = |\lambda n_3|/\Lambda \qquad \sinh\xi_n = (\lambda n_1 + 1)/\Lambda \tag{132}$$

then $\boldsymbol{n}_f$ has the form in equation (113), but $\epsilon = \epsilon(\lambda n_3)$. The solution is

$$E(t) = T(\varphi) S(\xi_n, \epsilon) W_+^{(b)}(\varphi) S^{-1}(\xi_n, \epsilon) \qquad E_q(t) = T_q(\varphi) W_{q+}^{(b)}(\varphi) \tag{133}$$

where $W_+^{(b)}(\varphi)$ and $W_{q+}^{(b)}(\varphi)$ have been given in equation (115). In this case the motion is oscillating with exponentially increasing amplitude.

The eigenvector of $E(t)$ corresponding to the eigenvalue 1 is (not normalized)

$$\begin{aligned}\boldsymbol{\eta}(t) = (&\epsilon(\sinh\xi_n \cosh\varphi \sin\Lambda\varphi - \sinh\varphi\cos\Lambda\varphi),\\ &\cosh\xi_n \sinh\varphi \sin\Lambda\varphi, \cosh\xi_n \cosh\varphi \sin\Lambda\varphi).\end{aligned} \tag{134}$$

If $\sin\Lambda\varphi = 0$ but $\varphi \neq 0$ at time τ, then $\boldsymbol{\eta}^2(\tau) < 0$ and there are no cyclic solutions in the time interval $[0, \tau]$. In the general case, we have

$$\boldsymbol{\eta}^2(t) = \cosh^2\xi_n \sin^2\Lambda\varphi - (\cosh^2\xi_n \cosh^2\varphi - 1)\sin^2[\Lambda\varphi - \arctan(\tanh\varphi/\sinh\xi_n)]. \tag{135}$$

This may be either positive or negative. For large φ the second term is large as long as the quantity in the square bracket is not close to $N\pi$, then $\boldsymbol{\eta}^2(\tau) < 0$, and there is no cyclic solution. However, if $\varphi(\tau)$ is close to φ_0 where φ_0 is the root of $\tan\Lambda\varphi = \tanh\varphi/\sinh\xi_n$, the second term is very small and $\boldsymbol{\eta}^2(\tau) > 0$, then we have cyclic solutions in the time interval $[0, \tau]$. The above transcendental equation has infinitely many roots, therefore in any time interval $(\tau_0, +\infty)$ with $\tau_0 > 0$ we can always find some τ such that cyclic solutions are available in $[0, \tau]$. In the present case there exists no τ such that $\varphi(\tau) \neq 0$ and $E_q(\tau) = \pm 1$.

(2) If $\boldsymbol{n}^2 = 1$ and $\lambda \in (n_1 - |n_3|, n_1 + |n_3|)$, or $\boldsymbol{n}^2 = -1$ and $\lambda \in (-\infty, -n_1 - |n_3|) \cup (-n_1 + |n_3|, \infty)$, or $\boldsymbol{n}^2 = 0$ and $\lambda n_1 > -1/2$, then $(\lambda n_3)^2 < (\lambda n_1 + 1)^2$. In this case we define

$$\Lambda = \sqrt{(\lambda n_1 + 1)^2 - (\lambda n_3)^2} \qquad \cosh \xi_n = |\lambda n_1 + 1|/\Lambda \qquad \sinh \xi_n = \lambda n_3/\Lambda \tag{136}$$

then $\boldsymbol{n}_f$ has the form in equation (119), but $\epsilon = \epsilon(\lambda n_1 + 1)$. The solution is

$$E(t) = T(\varphi) S(\xi_n, \epsilon) W_-^{(b)}(\varphi) S^{-1}(\xi_n, \epsilon) \qquad E_q(t) = T_q(\varphi) W_{q-}^{(b)}(\varphi) \tag{137}$$

where $W_-^{(b)}(\varphi)$ and $W_{q-}^{(b)}(\varphi)$ have been given in equation (121). In this case the motion is exponentially infinite.

The eigenvector of $E(t)$ corresponding to the eigenvalue 1 is (not normalized)

$$\begin{aligned}\boldsymbol{\eta}(t) = (&\epsilon \cosh \xi_n \cosh \varphi \sinh \Lambda\varphi \\ &- \sinh \varphi \cosh \Lambda\varphi, \sinh \xi_n \sinh \varphi \sinh \Lambda\varphi, \sinh \xi_n \cosh \varphi \sinh \Lambda\varphi)\end{aligned} \tag{138}$$

which gives

$$\boldsymbol{\eta}^2(t) = \sinh^2 \xi_n \sinh^2 \Lambda\varphi - (\cosh \xi_n \cosh \varphi \sinh \Lambda\varphi - \epsilon \sinh \varphi \cosh \Lambda\varphi)^2. \tag{139}$$

If $\epsilon = -1$, it is obvious that the second term is larger and $\boldsymbol{\eta}^2(\tau) < 0$, and thus there is no cyclic solution in any time interval $[0, \tau]$. If $\epsilon = 1$, the above result can be recast in the form

$$\boldsymbol{\eta}^2(t) = \sinh^2 \xi_n \sinh^2 \Lambda\varphi - (\sinh^2 \xi_n \cosh^2 \varphi + 1) \sinh^2[\Lambda\varphi - \operatorname{arctanh} (\tanh \varphi / \cosh \xi_n)]. \tag{140}$$

In this case, cyclic solutions are possible. For example, when $\Lambda < 1$, the transcendental equation $\tanh \varphi = \cosh \xi_n \tanh \Lambda\varphi$ has one root φ_0. When $\varphi(\tau)$ is close to φ_0, the second term is very small and $\boldsymbol{\eta}^2(\tau) > 0$. Then we can have cyclic solutions in $[0, \tau]$. Another possible case is $\varphi(\tau) \ll 1$, which also leads to $\boldsymbol{\eta}^2(\tau) > 0$ if $\exp(|\xi_n|) > \Lambda$. Then we also have cyclic solutions in $[0, \tau]$. However, when $\varphi(\tau)$ is large, it is easy to see from equation (139) that $\boldsymbol{\eta}^2(\tau) < 0$ regardless of the parameters ξ_n and Λ. Then there is no cyclic solution in $[0, \tau]$. In other words, no initial state can return to itself at a large time. This is rather different from the last case. As before, there exists no τ such that $\varphi(\tau) \neq 0$ and $E_q(\tau) = \pm 1$.

(3) If $\boldsymbol{n}^2 = 1$ and $\lambda = n_1 \pm |n_3|$, or $\boldsymbol{n}^2 = -1$ and $\lambda = -n_1 \pm |n_3|$, or $\boldsymbol{n}^2 = 0$ and $\lambda n_1 = -1/2$, then $(\lambda n_3)^2 = (\lambda n_1 + 1)^2$. Let $\epsilon = (\lambda n_1 + 1)/\lambda n_3 = \pm 1$. The solution is

$$E(t) = T(\varphi) D^{(c)}(\varphi) \qquad E_q(t) = T_q(\varphi) W_{q0}^{(c)}(\varphi) \tag{141}$$

where $D^{(c)}(\varphi)$ and $W_{q0}^{(c)}(\varphi)$ can be obtained from $D^{(b)}(\varphi)$ and $W_{q0}^{(b)}(\varphi)$ in equation (125), respectively, by substituting $\epsilon\lambda n_3$ for λn_1. In this case the motion is also exponentially infinite, but different from the last case. Indeed, both $W_-^{(b)}(\varphi)$ and $W_{q-}^{(b)}(\varphi)$ in equation (137) involve exponential functions of φ, while $D^{(c)}(\varphi)$ and $W_{q0}^{(c)}(\varphi)$ involve polynomials of it.

The eigenvector of $E(t)$ corresponding to the eigenvalue 1 is (not normalized)

$$\boldsymbol{\eta}(t) = (\epsilon\lambda n_3 \varphi \cosh \varphi - \sinh \varphi, \lambda n_3 \varphi \sinh \varphi, \lambda n_3 \varphi \cosh \varphi) \tag{142}$$

which gives

$$\boldsymbol{\eta}^2(t) = (\lambda n_3 \varphi)^2 - (\epsilon\lambda n_3 \varphi \cosh \varphi - \sinh \varphi)^2. \tag{143}$$

If $\epsilon\lambda n_3 < 0$, it is obvious that the second term is larger and $\boldsymbol{\eta}^2(\tau) < 0$, and thus there is no cyclic solution in any time interval $[0, \tau]$. If $\epsilon\lambda n_3 > 0$, cyclic solutions are possible. For example, when $|\lambda n_3| < 1$, the transcendental equation $\tanh \varphi = \epsilon\lambda n_3 \varphi$ has one nonzero root φ_0. When $\varphi(\tau)$ is close to φ_0, the second term is very small and $\boldsymbol{\eta}^2(\tau) > 0$. Then we can have cyclic solutions in $[0, \tau]$. Another possible case is $\varphi(\tau) \ll 1$, which also leads to $\boldsymbol{\eta}^2(\tau) > 0$ and we have cyclic solutions in $[0, \tau]$. However, when $\varphi(\tau)$ is large, it is easy to see that

$\boldsymbol{\eta}^2(\tau) < 0$ regardless of the parameters n_3 and λ. Then there is no cyclic solution in $[0, \tau]$. In other words, no initial state can return to itself at a large time. This is similar to the last case. As before, there exists no τ such that $\varphi(\tau) \neq 0$ and $E_q(\tau) = \pm 1$.

In this subsection we also see that different values of λ correspond to different patterns of motion, for a Hamiltonian of a definite nature. But the cases are rather different from those in subsection 6.2. In some of the cases here, there is no cyclic solution in any time interval.

6.4. $\boldsymbol{n} = (n_1, n_3 \cosh 2\varphi, n_3 \sinh 2\varphi)$

Here n_1 and n_3 are constants, satisfying $\boldsymbol{n}^2 = -n_3^2 - n_1^2 = -1$. This is different from the previous cases, since $\boldsymbol{n}^2$ cannot take 1 and 0. $\varphi = \varphi(t)$ has the same property as before. If $\omega(t) = \lambda\dot{\varphi}(t)$ where λ is a constant, analytic solutions are available. The time evolution operator is given by equation (128) but now

$$\Lambda \boldsymbol{n}_f = (\lambda n_1 + 1, \lambda n_3, 0). \tag{144}$$

The equation for $\boldsymbol{e}(t)$ can be solved in a way similar to that in subsection 6.3. We define

$$\Lambda = \sqrt{(\lambda n_1 + 1)^2 + (\lambda n_3)^2} \qquad \cos 2\varphi_n = (\lambda n_1 + 1)/\Lambda \qquad \sin 2\varphi_n = \lambda n_3/\Lambda \tag{145}$$

then $\boldsymbol{n}_f = (\cos 2\varphi_n, \sin 2\varphi_n, 0)$. The solution is

$$E(t) = T(\varphi)R(\varphi_n)W_-^{(b)}(\varphi)R^{-1}(\varphi_n) \qquad E_q(t) = T_q(\varphi)R_q(\varphi_n)W_q^{(d)}(\varphi)R_q^{-1}(\varphi_n) \tag{146}$$

where $W_-^{(b)}(\varphi)$ is given in equation (121) and

$$W_q^{(d)}(\varphi) = \begin{pmatrix} \cosh \Lambda\varphi & \sinh \Lambda\varphi \\ \sinh \Lambda\varphi & \cosh \Lambda\varphi \end{pmatrix}. \tag{147}$$

Obviously, the motion is exponentially infinite.

The eigenvector of $E(t)$ corresponding to the eigenvalue 1 is (not normalized)

$$\boldsymbol{\eta}(t) = (\cos 2\varphi_n \cosh\varphi \sinh \Lambda\varphi - \sinh\varphi \cosh \Lambda\varphi, \sin 2\varphi_n \cosh\varphi \sinh \Lambda\varphi, \sin 2\varphi_n \sinh\varphi \sinh \Lambda\varphi) \tag{148}$$

which gives

$$\boldsymbol{\eta}^2(t) = -\sin^2 2\varphi_n \sinh^2 \Lambda\varphi - (\cos 2\varphi_n \cosh\varphi \sinh \Lambda\varphi - \sinh\varphi \cosh \Lambda\varphi)^2. \tag{149}$$

It is obvious that $\boldsymbol{\eta}^2(\tau) < 0$ if $\varphi(\tau) \neq 0$, and there is no cyclic solution in any time interval $[0, \tau]$. In other words, in this case no initial state can return to itself at a later time. As before, there exists no τ such that $\varphi(\tau) \neq 0$ and $E_q(\tau) = \pm 1$.

7. Summary

In this paper we develop a method for solving the Schrödinger equation of the generalized time-dependent harmonic oscillator. This method, though not always convenient for practical calculation of the time evolution operator, is very suitable for the study of cyclic solutions and geometric phases. We concentrate our attention on Hamiltonians of general time dependence and cyclic solutions in the time interval $[0, \tau]$ with an arbitrarily given τ. A necessary and sufficient condition for the existence of cyclic solutions in such time intervals is given. There may exist some time interval in which more solutions are cyclic. This includes several cases among which two are of more interest. In one of these cases all solutions are cyclic, and in another all solutions with definite parity are cyclic. Criteria for the appearance of such cases are given. The proportional relation between the nonadiabatic geometric phase and the classical

Hannay angle is reestablished. However, this holds only for cyclic solutions with special initial conditions. For more general cyclic solutions encountered in the above cases, the nonadiabatic geometric phase contains in general an extra term in addition to the one proportional to the classical Hannay angle. Several examples are studied where the Hamiltonians are relatively simple and analytic solutions are available. In these examples the existence of cyclic solutions is discussed in detail. Many possibilities are exhibited: (1) cyclic solutions are available for all τ; (2) cyclic solutions available for some τ and such τ may be found in any interval $(\tau_0, +\infty)$ with $\tau_0 > 0$; (3) cyclic solutions available for some τ but such τ exist only in some finite interval $(0, \tau_0)$; (4) cyclic solutions are not available at all. From the point of view of wave packets, the motion in these examples also exhibits various patterns. For a Hamiltonian of a definite nature, say, elliptic, several different patterns of motion are possible, depending on the value of some parameter in the Hamiltonian. There exists some critical value of the parameter at which some kind of phase transition happens. When the parameter goes through it, the motion changes from one pattern to another, and at the critical value itself the motion has an independent pattern.

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Appendix

Our approach to the solution of the Schrödinger equation with time-dependent Hamiltonians was first used in our previous paper [36] for the system of general spin moving in an arbitrarily varying magnetic field, and is further developed in this paper. The Hamiltonian for the spin in a magnetic field is an element of the $SO(3)$ algebra, while that for the time-dependent harmonic oscillator is one of the $SO(2, 1)$ algebra. In this appendix we briefly discuss how to extend the formalism to a system where the Hamiltonian is an element of a more general Lie algebra.

Consider a Lie algebra with the generators K_a $(a = 1, 2, \ldots, l)$, satisfying the commutation relation

$$[K_a, K_b] = \mathrm{i} f_{ab}{}^c K_c \tag{A.1}$$

where $f_{ab}{}^c$ are the structure constants. Define

$$g_{ab} = f_{ac}{}^d f_{bd}{}^c = \mathrm{tr}\,(f_a f_b) \tag{A.2}$$

where f_a is a matrix whose element is defined by $(f_a)_b{}^c = f_{ab}{}^c$. It is obviously symmetric: $g_{ba} = g_{ab}$. For a semisimple Lie algebra, the matrix g_{ab} is not singular, and thus invertible. Its inverse matrix is denoted by g^{ab}, satisfying $g^{ac} g_{cb} = \delta^a{}_b$. The matrices g_{ab} and g^{ab} act like metric tensors and can be used to lower and raise vector indices. For the $SO(2, 1)$ algebra with the generators given in equation (2), one should find $g_{ab} = 8\,\mathrm{diag}\,(1, 1, -1)$ and $g^{ab} = \frac{1}{8}\,\mathrm{diag}\,(1, 1, -1)$. Thus the metric used in the main text is different from this by a constant factor.

For simplicity we consider a semisimple Lie algebra of rank 2, say, the $SU(3)$ algebra. It contains two mutually commuting operators which generate the Cartan subalgebra. Thus there are two Casimir operators, given by

$$C_2 = g^{ab} K_a K_b \qquad C_3 = h^{abc} K_a K_b K_c \tag{A.3}$$

where h^{abc} is obtained by raising the indices of h_{abc}, and the latter is defined by

$$h_{abc} = \mathrm{tr}\,(f_a f_b f_c). \tag{A.4}$$

Now consider a physical system whose Hamiltonian is an element of the above semisimple Lie algebra. More specifically,

$$H(t) = \hbar\omega^a(t)K_a \tag{A.5}$$

where $\omega^a(t)$ are time-dependent frequency parameters, and the generators K_a are dimensionless. We define a l-component vector $\boldsymbol{u}(t) = (u_1(t), u_2(t), \ldots, u_l(t))$ by

$$u_a(t) = \langle K_a \rangle \equiv (\psi(t), K_a\psi(t)) \tag{A.6}$$

where $\psi(t)$ is an arbitrary state of the system, that is, a solution to the Schrödinger equation. It is easy to show that it satisfies the equation

$$\dot{u}_a(t) = f_{ab}{}^c\omega^b(t)u_c(t). \tag{A.7}$$

This is a system of l linear differential equations. Now we define a l-component vector $\boldsymbol{e}(t) = (e_1(t), e_2(t), \ldots, e_l(t))$ by the same equation, that is

$$\dot{e}_a(t) = f_{ab}{}^c\omega^b(t)e_c(t) \tag{A.8}$$

and a nontrivial (nonzero) initial condition. It is then straightforward to show that the two operators linear in K_a, defined by

$$L_1(t) = g^{ab}e_a(t)K_b \qquad L_2(t) = h^{abc}e_a(t)e_b(t)K_c \tag{A.9}$$

are invariant operators, and they commute with each other. Therefore, the invariant operators can be obtained by solving a linear differential equation for $\boldsymbol{e}(t)$. It can be similarly shown that $g^{ab}e_a(t)e_b(t)$ and $h^{abc}e_a(t)e_b(t)e_c(t)$ are also time-independent quantities. Thus among the l components of $\boldsymbol{e}(t)$ only $l-2$ are independent variables. We can parametrize $\boldsymbol{e}(t)$ in some way similar to that in equation (15). The independent parameters will be denoted by $\xi(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_{l-2}(t))$.

The subsequent steps in obtaining the time evolution operator depend on the details of the Lie algebra. We can give only a rough sketch here.

The crucial step is to find a unitary operator $Q(t) = Q(\xi(t))$ such that

$$L_1(t) = Q(t)H_1Q^\dagger(t) \qquad L_2(t) = Q(t)H_2Q^\dagger(t) \tag{A.10}$$

where H_1 and H_2 are generators of the Cartan subalgebra. It might be difficult to prove the existence of $Q(t)$ in the general case. For a specific Lie algebra, however, one can try to find it by practical calculations like those in section 2. Since H_1 and H_2 commute with each other, they have a complete set of common eigenstates. These will be denoted by $\{\psi_{nm}\}$, satisfying

$$H_1\psi_{nm} = \lambda_n\psi_{nm} \qquad H_2\psi_{nm} = \mu_m\psi_{nm}. \tag{A.11}$$

Obviously, λ_n and μ_m are also the eigenvalues of $L_1(t)$ and $L_2(t)$, respectively.

Next we will show that if the initial state $\psi(0)$ of the system satisfies

$$L_1(0)\psi(0) = \lambda_n\psi(0) \qquad L_2(0)\psi(0) = \mu_m\psi(0) \tag{A.12}$$

then the state $\psi(t)$ at later times will satisfy

$$L_1(t)\psi(t) = \lambda_n\psi(t) \qquad L_2(t)\psi(t) = \mu_m\psi(t). \tag{A.13}$$

We use here a different proof from the induction one employed in section 3. It is easy to show that $\psi^{(1)}(t) \equiv [L_1(t) - \lambda_n]\psi(t)$ satisfies

$$\mathrm{i}\hbar\partial_t\psi^{(1)}(t) = H(t)\psi^{(1)}(t) \tag{A.14}$$

where the equation $i\hbar\partial_t L_1(t) + [L_1(t), H(t)] = 0$ has been used. Consequently, $\psi^{(1)}(t) = U(t)\psi^{(1)}(0)$, where $U(t)$ is the time evolution operator of the Schrödinger equation. Because equation (A.12) leads to $\psi^{(1)}(0) = 0$, we have $\psi^{(1)}(t) = 0$, and thus the first equation in (A.13) is obtained. The second one can be proved in a similar way.

On account of the above results, we have

$$\psi(t) = \exp[i\alpha_{nm}(t)]Q(t)\psi_{nm} \tag{A.15}$$

where $\alpha_{nm}(t)$ is a phase that cannot be determined by the eigenvalue equation. However, it is not arbitrary. By the requirement that $\psi(t)$ satisfies the Schrödinger equation, it can be determined in terms of $\xi(t)$. Finally, one should manage to replace λ_n appearing in $\alpha_{nm}(t)$ and $\alpha_{nm}(0)$ by H_1, and μ_m by H_2, and obtain

$$\psi(t) = Q(t)\exp[i\alpha(H_1, H_2, t)]Q^\dagger(0)\psi(0). \tag{A.16}$$

Because all the operators in the above equation are independent of n and m, and because the set $\{\psi_{nm}\}$ is complete, we obtain the time evolution operator

$$U(t) = Q(t)\exp[i\alpha(H_1, H_2, t)]Q^\dagger(0). \tag{A.17}$$

The time dependence in both $Q(t)$ and $\alpha(H_1, H_2, t)$ comes from $\xi(t)$ (and $\omega^a(t)$ of course). Therefore, the time evolution operator is also obtained by solving the linear different equation for $\boldsymbol{e}(t)$.

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